On the Weak Continuation of Reverse Bisimilarity vs. Forward Bisimilarity

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Abstract. We introduce a minimal process calculus for nondeterministic systems that are reversible, i.e., capable of undoing their actions starting from the last performed one. The considered systems are sequential so as to be neutral with respect to interleaving semantics vs. truly concurrent semantics. As a natural continuation of previous work on strong bisimilarity in this reversible setting, we investigate compositionality properties and equational characterizations of weak variants of forward-reverse bisimilarity as well as of its two components, i.e., weak forward bisimilarity and weak reverse bisimilarity.

1 Introduction

Reversibility in computing started to gain attention since the seminal works [8,1], where it was shown that reversible computations may achieve low levels of heat dissipation. Nowadays *reversible computing* has many applications ranging from computational biochemistry and parallel discrete-event simulation to robotics, control theory, fault tolerant systems, and concurrent program debugging.

In a reversible system, we can observe two directions of computation: a *forward* one, coinciding with the normal way of computing, and a *backward* one, along which the effects of the forward one are undone when needed in a *causally consistent* way, i.e., by returning to a past consistent state. The latter task is not easy to accomplish in a concurrent system, because the undo procedure necessarily starts from the last performed action and this may not be unique. The usually adopted strategy is that an action can be undone provided that all of its consequences, if any, have been undone beforehand [3].

In the process algebra literature, two approaches have been developed to reverse computations based on keeping track of past actions: the dynamic one of [3] and the static one of [13], later shown to be equivalent in terms of labeled transition systems isomorphism [9].

The former yields RCCS, a variant of CCS [11] that uses stack-based memories attached to processes to record all the actions executed by those processes. A single transition relation is defined, while actions are divided into forward and backward resulting in forward and backward transitions. This approach is suitable when the operational semantics is given in terms of reduction semantics, like in the case of very expressive calculi as well as programming languages.

In contrast, the latter proposes a general method, of which CCSK is a result, to reverse calculi, relying on the idea of retaining within the process syntax all executed actions, which are suitably decorated, and dynamic operators, which are thus made static. A forward transition relation and a backward transition relation are separately defined, which are labeled with actions extended with communication keys so as to remember who synchronized with whom when going backward. This approach is very handy when it comes to deal with labeled transition systems and basic process calculi.

In [13] *forward-reverse bisimilarity* was introduced too. Unlike standard bisimilarity [12,11], it is truly concurrent as it does not satisfy the expansion law of parallel composition into a choice among all possible action sequencings. The interleaving view can be restored in a reversible setting by employing *back-and-forth bisimilarity* [4]. This is defined on computation paths instead of states, thus preserving not only causality but also history as backward moves are constrained to take place along the path followed when going forward even in the presence of concurrency. In the latter setting, a single transition relation is considered, which is viewed as bidirectional, and in the bisimulation game the distinction between going forward or backward is made by matching outgoing or incoming transitions of the considered processes, respectively.

In [2] forward-reverse bisimilarity and its two components, i.e., forward bisimilarity and reverse bisimilarity, have been investigated in terms of compositionality properties and equational characterizations both for nondeterministic processes and Markovian processes. In order to remain neutral with respect to interleaving view vs. true concurrency, the study has been conducted over a sequential processes calculus, in which parallel composition is not admitted so that not even the communication keys of [13] are needed. Furthermore, a single transition relation and the distinction between outgoing and incoming transitions have been exploited like in [4].

In this paper we extend the work done in [2] to *weak* variants of forward-reverse, forward, and reverse bisimilarities over nondeterministic reversible sequential processes, where by weak we mean that the considered equivalences abstract from unobservable actions, traditionally denoted by τ. As far as compositionality is concerned, compared to [2] we discover that an initiality condition is necessary not only for forward bisimilarity but also for forward-reverse bisimilarity, which additionally solves the congruence problem with respect to nondeterministic choice affecting all weak variants of bisimilarity [11,6]. As for equational characterizations, we retrieve the τ-laws of weak bisimilarity [11] and branching bisimilarity [6] over standard forward-only processes, along with some variants of those laws addressing the backward direction.

The paper is organized as follows. In Section 2 we recall syntax and semantics for the calculus of nondeterministic reversible sequential processes as well as the forward, reverse, and forward-reverse bisimilarities introduced in [2]. In Section 3 we define the weak variants of the three aforementioned bisimilarities. In Section 4 we study their compositionality properties. Finally, in Section 5 we provide sound and ground-complete equational characterizations for weak forward bisimilarity and weak reverse bisimilarity, together with a sound equational characterization for weak forward-reverse bisimilarity.

2 Background

2.1 Syntax of Nondeterministic Reversible Sequential Processes

Given a countable set A of actions – ranged over by a, b, c – including an unobservable action denoted by τ, the syntax of reversible sequential processes is as follows [2]:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P$$

where:

- $\underline{0}$ is the terminated process.
- $a \,.\, P$ is a process that can execute action a and whose continuation is P.
- $a^{\dagger} \,.\, P$ is a process that executed action a and whose continuation is in P.
- $P_1 + P_2$ expresses a nondeterministic choice between P_1 and P_2 as far as both of them have not executed any action yet.

We syntactically characterize through suitable predicates three classes of processes generated by the grammar above. Firstly, we have *initial* processes, i.e., processes in which all the actions are unexecuted:

$$\begin{aligned} &\mathit{initial}(\underline{0}) \\ &\mathit{initial}(a \,.\, P) \Longleftarrow \mathit{initial}(P) \\ &\mathit{initial}(P_1 + P_2) \Longleftarrow \mathit{initial}(P_1) \wedge \mathit{initial}(P_2) \end{aligned}$$

Secondly, we have *final* processes, i.e., processes in which all the actions along a single path have been executed:

$$\begin{aligned} &\mathit{final}(\underline{0}) \\ &\mathit{final}(a^{\dagger} .\, P) \Longleftarrow \mathit{final}(P) \\ &\mathit{final}(P_1 + P_2) \Longleftarrow (\mathit{final}(P_1) \wedge \mathit{initial}(P_2)) \vee \\ &\qquad\qquad\qquad\qquad (\mathit{initial}(P_1) \wedge \mathit{final}(P_2)) \end{aligned}$$

Multiple paths arise only in the presence of alternative compositions, i.e., nondeterministic choices. At each occurrence of +, only the subprocess chosen for execution can move, while the other one, although not selected, is kept as an initial subprocess within the overall process to support reversibility.

Thirdly, we have the processes that are *reachable* from an initial one, whose set we denote by $\mathbb{P}$:

$$\begin{aligned} &\mathit{reachable}(\underline{0}) \\ &\mathit{reachable}(a \,.\, P) \Longleftarrow \mathit{initial}(P) \\ &\mathit{reachable}(a^{\dagger} .\, P) \Longleftarrow \mathit{reachable}(P) \\ &\mathit{reachable}(P_1 + P_2) \Longleftarrow (\mathit{reachable}(P_1) \wedge \mathit{initial}(P_2)) \vee \\ &\qquad\qquad\qquad\qquad (\mathit{initial}(P_1) \wedge \mathit{reachable}(P_2)) \end{aligned}$$

It is worth noting that:

- $\underline{0}$ is the only process that is both initial and final as well as reachable.
- Any initial or final process is reachable too.
- $\mathbb{P}$ also contains processes that are neither initial nor final, like e.g. $a^{\dagger} .\, b \,.\, \underline{0}$.
- The relative positions of already executed actions and actions to be executed matter; in particular, an action of the former kind can never follow one of the latter kind. For instance, $a^{\dagger} .\, b \,.\, \underline{0} \in \mathbb{P}$ whereas $b \,.\, a^{\dagger} .\, \underline{0} \notin \mathbb{P}$.

$(\textsc{Act}_{\text{f}})\ \dfrac{initial(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P}$	$(\textsc{Act}_{\text{p}})\ \dfrac{P \xrightarrow{b} P'}{a^{\dagger}\,.\,P \xrightarrow{b} a^{\dagger}\,.\,P'}$
$(\textsc{Cho}_{\text{l}})\ \dfrac{P_1 \xrightarrow{a} P'_1 \quad initial(P_2)}{P_1 + P_2 \xrightarrow{a} P'_1 + P_2}$	$(\textsc{Cho}_{\text{r}})\ \dfrac{P_2 \xrightarrow{a} P'_2 \quad initial(P_1)}{P_1 + P_2 \xrightarrow{a} P_1 + P'_2}$

Table 1. Operational semantic rules for reversible action prefix and choice

2.2 Operational Semantic Rules

According to the approach of [13], dynamic operators such as action prefix and alternative composition have to be made static by the semantics, so as to retain within the syntax all the information needed to enable reversibility. For the sake of minimality, unlike [13] we do not generate two distinct transition relations – a forward one $\longrightarrow$ and a backward one $\leadsto$ – but a single transition relation, which we implicitly regard as being symmetric like in [4] to enforce the *loop property*: each executed action can be undone and each undone action can be redone.

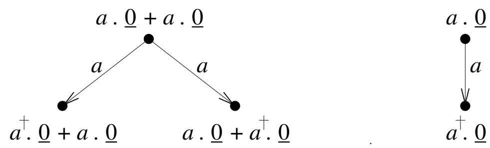
In our setting, a backward transition from P' to P ($P' \overset{a}{\leadsto} P$) is subsumed by the corresponding forward transition t from P to P' ($P \xrightarrow{a} P'$). As will become clear with the definition of behavioral equivalences, like in [4] when going forward we view t as an *outgoing* transition of P, while when going backward we view t as an *incoming* transition of P'. The semantic rules for $\longrightarrow \subseteq \mathbb{P} \times A \times \mathbb{P}$ are defined in Table 1 and generate the labeled transition system $(\mathbb{P}, A, \longrightarrow)$ [2].

The first rule for action prefix ($\textsc{Act}_{\text{f}}$ where f stands for forward) applies only if P is initial and retains the executed action in the target process of the generated forward transition by decorating the action itself with †. The second rule for action prefix ($\textsc{Act}_{\text{p}}$ where p stands for propagation) propagates actions executed by inner initial subprocesses.

In both rules for alternative composition ($\textsc{Cho}_{\text{l}}$ and $\textsc{Cho}_{\text{r}}$ where l stands for left and r stands for right), the subprocess that has not been selected for execution is retained as an initial subprocess in the target process of the generated transition. When both subprocesses are initial, both rules for alternative composition are applicable, otherwise only one of them can be applied and in that case it is the non-initial subprocess that can move, because the other one has been discarded at the moment of the selection.

Every state corresponding to a non-final process has at least one outgoing transition, while every state corresponding to a non-initial process has exactly one incoming transition due to the decoration of executed actions. The labeled transition system underlying an initial process turns out to be a tree, whose branching points correspond to occurrences of $+$.

Example 1. The labeled transition systems generated by the rules in Table 1 for the two initial processes $a\,.\,\underline{0} + a\,.\,\underline{0}$ and $a\,.\,\underline{0}$ are depicted below:



As far as the one on the left is concerned, we observe that, in the case of a standard process calculus, a single a-transition from $a\,.\,\underline{0} + a\,.\,\underline{0}$ to $\underline{0}$ would have been generated due to the absence of action decorations within processes. ■

2.3 Strong Forward, Reverse, and Forward-Reverse Bisimilarities

While forward bisimilarity considers only *outgoing* transitions [12,11], reverse bisimilarity considers only *incoming* transitions. Forward-reverse bisimilarity [13] considers instead both outgoing transitions and incoming ones. Here are their *strong* versions studied in [2], where strong means not abstracting from τ-actions.

Definition 1. *We say that* $P_1, P_2 \in \mathbb{P}$ *are* forward bisimilar*, written* $P_1 \sim_{\mathrm{FB}} P_2$*, iff* $(P_1, P_2) \in \mathcal{B}$ *for some forward bisimulation* $\mathcal{B}$*. A symmetric relation* $\mathcal{B}$ *over* $\mathbb{P}$ *is a* forward bisimulation *iff for all* $(P_1, P_2) \in \mathcal{B}$ *and* $a \in A$*:*

- *Whenever* $P_1 \xrightarrow{a} P_1'$*, then* $P_2 \xrightarrow{a} P_2'$ *with* $(P_1', P_2') \in \mathcal{B}$*.* ■

Definition 2. *We say that* $P_1, P_2 \in \mathbb{P}$ *are* reverse bisimilar*, written* $P_1 \sim_{\mathrm{RB}} P_2$*, iff* $(P_1, P_2) \in \mathcal{B}$ *for some reverse bisimulation* $\mathcal{B}$*. A symmetric relation* $\mathcal{B}$ *over* $\mathbb{P}$ *is a* reverse bisimulation *iff for all* $(P_1, P_2) \in \mathcal{B}$ *and* $a \in A$*:*

- *Whenever* $P_1' \xrightarrow{a} P_1$*, then* $P_2' \xrightarrow{a} P_2$ *with* $(P_1', P_2') \in \mathcal{B}$*.* ■

Definition 3. *We say that* $P_1, P_2 \in \mathbb{P}$ *are* forward-reverse bisimilar*, written* $P_1 \sim_{\mathrm{FRB}} P_2$*, iff* $(P_1, P_2) \in \mathcal{B}$ *for some forward-reverse bisimulation* $\mathcal{B}$*. A symmetric relation* $\mathcal{B}$ *over* $\mathbb{P}$ *is a* forward-reverse bisimulation *iff for all* $(P_1, P_2) \in \mathcal{B}$ *and* $a \in A$*:*

- *Whenever* $P_1 \xrightarrow{a} P_1'$*, then* $P_2 \xrightarrow{a} P_2'$ *with* $(P_1', P_2') \in \mathcal{B}$*.*
- *Whenever* $P_1' \xrightarrow{a} P_1$*, then* $P_2' \xrightarrow{a} P_2$ *with* $(P_1', P_2') \in \mathcal{B}$*.* ■

$\sim_{\mathrm{FRB}} \subsetneq \sim_{\mathrm{FB}} \cap \sim_{\mathrm{RB}}$ with the inclusion being strict because, e.g., the two final processes $a^{\dagger}\,.\,\underline{0}$ and $a^{\dagger}\,.\,\underline{0} + c\,.\,\underline{0}$ are identified by $\sim_{\mathrm{FB}}$ (no outgoing transitions on both sides) and by $\sim_{\mathrm{RB}}$ (only an incoming a-transition on both sides), but distinguished by $\sim_{\mathrm{FRB}}$ as in the latter process action c is enabled again after undoing a (and hence there is an outgoing c-transition in addition to an outgoing a-transition). Moreover, $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$ are incomparable because for instance:

$$a^{\dagger}\,.\,\underline{0} \sim_{\mathrm{FB}} \underline{0} \;\text{ but }\; a^{\dagger}\,.\,\underline{0} \not\sim_{\mathrm{RB}} \underline{0}$$
$$a\,.\,\underline{0} \sim_{\mathrm{RB}} \underline{0} \;\text{ but }\; a\,.\,\underline{0} \not\sim_{\mathrm{FB}} \underline{0}$$

Note that that $\sim_{\mathrm{FRB}} = \sim_{\mathrm{FB}}$ over initial processes, with $\sim_{\mathrm{RB}}$ strictly coarser, whilst $\sim_{\mathrm{FRB}} \neq \sim_{\mathrm{RB}}$ over final processes because, after going backward, previously discarded subprocesses come into play again in the forward direction.

Example 2. The two processes considered in Example 1 are identified by all the three equivalences. This is witnessed by any bisimulation that contains the pairs $(a\,.\,\underline{0} + a\,.\,\underline{0}, a\,.\,\underline{0})$, $(a^{\dagger}.\,\underline{0} + a\,.\,\underline{0}, a^{\dagger}.\,\underline{0})$, and $(a\,.\,\underline{0} + a^{\dagger}.\,\underline{0}, a^{\dagger}.\,\underline{0})$. ■

As observed in [2], it makes sense that $\sim_{\rm FB}$ identifies processes with a different past and that $\sim_{\rm RB}$ identifies processes with a different future, in particular with $\underline{0}$ that has neither past nor future. However, for $\sim_{\rm FB}$ this results in a compositionality violation with respect to alternative composition. As an example:

$$a^{\dagger}.\,b\,.\,\underline{0} \;\sim_{\rm FB}\; b\,.\,\underline{0}$$
$$a^{\dagger}.\,b\,.\,\underline{0} + c\,.\,\underline{0} \;\not\sim_{\rm FB}\; b\,.\,\underline{0} + c\,.\,\underline{0}$$

because in $a^{\dagger}.\,b\,.\,\underline{0} + c\,.\,\underline{0}$ action c is disabled due to the presence of the already executed action $a^{\dagger}$, while in $b\,.\,\underline{0} + c\,.\,\underline{0}$ action c is enabled as there are no past actions preventing it from occurring. Note that a similar phenomenon does not happen with $\sim_{\rm RB}$ as $a^{\dagger}.\,b\,.\,\underline{0} \not\sim_{\rm RB} b\,.\,\underline{0}$ due to the incoming a-transition of $a^{\dagger}.\,b\,.\,\underline{0}$.

This problem, which does not show up for $\sim_{\rm RB}$ and $\sim_{\rm FRB}$ because these two equivalences cannot identify an initial process with a non-initial one, leads to the following variant of $\sim_{\rm FB}$ that is sensitive to the presence of the past.

Definition 4. *We say that $P_1, P_2 \in \mathbb{P}$ are* past-sensitive forward bisimilar*, written $P_1 \sim_{\rm FB:ps} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some past-sensitive forward bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a* past-sensitive forward bisimulation *iff for all $(P_1, P_2) \in \mathcal{B}$:*

- *$initial(P_1) \Longleftrightarrow initial(P_2)$.*
- *For all $a \in A$, whenever $P_1 \xrightarrow{a} P_1'$, then $P_2 \xrightarrow{a} P_2'$ with $(P_1', P_2') \in \mathcal{B}$.* ■

Now $\sim_{\rm FB:ps}$ is sensitive to the presence of the past:

$$a^{\dagger}.\,b\,.\,\underline{0} \;\not\sim_{\rm FB:ps}\; b\,.\,\underline{0}$$

but can still identify non-initial processes having a different past:

$$a_1^{\dagger}\,.\,P \;\sim_{\rm FB:ps}\; a_2^{\dagger}\,.\,P$$

It holds that $\sim_{\rm FRB} \subsetneq \sim_{\rm FB:ps} \cap \sim_{\rm RB}$, with $\sim_{\rm FRB} = \sim_{\rm FB:ps}$ over initial processes as well as $\sim_{\rm FB:ps}$ and $\sim_{\rm RB}$ being incomparable because, e.g., for $a_1 \neq a_2$:

$$a_1^{\dagger}\,.\,P \;\sim_{\rm FB:ps}\; a_2^{\dagger}\,.\,P \;\text{ but }\; a_1^{\dagger}\,.\,P \;\not\sim_{\rm RB}\; a_2^{\dagger}\,.\,P$$
$$a_1\,.\,P \;\sim_{\rm RB}\; a_2\,.\,P \;\text{ but }\; a_1\,.\,P \;\not\sim_{\rm FB:ps}\; a_2\,.\,P$$

In [2] it has been shown that all the considered bisimilarities are congruences with respect to action prefix, while only $\sim_{\rm FB:ps}$, $\sim_{\rm RB}$, and $\sim_{\rm FRB}$ are congruences with respect to alternative composition too, with $\sim_{\rm FB:ps}$ being the coarsest congruence with respect to $+$ contained in $\sim_{\rm FB}$. Sound and ground-complete equational characterizations have also been provided for the three congruences.

3 Weak Bisimilarity and Reversibility

In this section we introduce *weak* variants of forward, reverse, and forward-reverse bisimilarities, i.e., variants capable of abstracting from τ-actions.

In the following definitions, $P \stackrel{\tau^*}{\Longrightarrow} P'$ means that $P' = P$ or there exists a nonempty sequence of finitely many τ-transitions such that the target of each of them coincides with the source of the subsequent one, with the source of the first one being P and the target of the last one being P'. Moreover, $\stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow}$ stands for an a-transition possibly preceded and followed by finitely many τ-transitions. We further let $\bar{A} = A \setminus \{\tau\}$.

Definition 5. *We say that $P_1, P_2 \in \mathbb{P}$ are* weakly forward bisimilar*, written $P_1 \approx_{\rm FB} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak forward bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a* weak forward bisimulation *iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:*

- *Whenever $P_1 \stackrel{\tau}{\longrightarrow} P_1'$, then $P_2 \stackrel{\tau^*}{\Longrightarrow} P_2'$ and $(P_1', P_2') \in \mathcal{B}$.*
- *Whenever $P_1 \stackrel{a}{\longrightarrow} P_1'$ for $a \in \bar{A}$, then $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2'$ and $(P_1', P_2') \in \mathcal{B}$.* ■

Definition 6. *We say that $P_1, P_2 \in \mathbb{P}$ are* weakly reverse bisimilar*, written $P_1 \approx_{\rm RB} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak reverse bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a* weak reverse bisimulation *iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:*

- *Whenever $P_1' \stackrel{\tau}{\longrightarrow} P_1$, then $P_2' \stackrel{\tau^*}{\Longrightarrow} P_2$ and $(P_1', P_2') \in \mathcal{B}$.*
- *Whenever $P_1' \stackrel{a}{\longrightarrow} P_1$ for $a \in \bar{A}$, then $P_2' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2$ and $(P_1', P_2') \in \mathcal{B}$.* ■

Definition 7. *We say that $P_1, P_2 \in \mathbb{P}$ are* weakly forward-reverse bisimilar*, written $P_1 \approx_{\rm FRB} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak forward-reverse bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a* weak forward-reverse bisimulation *iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:*

- *Whenever $P_1 \stackrel{\tau}{\longrightarrow} P_1'$, then $P_2 \stackrel{\tau^*}{\Longrightarrow} P_2'$ and $(P_1', P_2') \in \mathcal{B}$.*
- *Whenever $P_1 \stackrel{a}{\longrightarrow} P_1'$ for $a \in \bar{A}$, then $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2'$ and $(P_1', P_2') \in \mathcal{B}$.*
- *Whenever $P_1' \stackrel{\tau}{\longrightarrow} P_1$, then $P_2' \stackrel{\tau^*}{\Longrightarrow} P_2$ and $(P_1', P_2') \in \mathcal{B}$.*
- *Whenever $P_1' \stackrel{a}{\longrightarrow} P_1$ for $a \in \bar{A}$, then $P_2' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2$ and $(P_1', P_2') \in \mathcal{B}$.* ■

Similar to their strong counterparts, it holds that $\approx_{\rm FRB} \subsetneq \approx_{\rm FB} \cap \approx_{\rm RB}$ with $\approx_{\rm FB}$ and $\approx_{\rm RB}$ being incomparable. Furthermore, each of the three weak bisimilarities is strictly coarser than the corresponding strong one.

4 Congruence Properties

In this section we investigate the compositionality of the three weak bisimilarities with respect to the considered process operators. Firstly, we observe that $\approx_{\rm FB}$ suffers from the same problem with respect to alternative composition as $\sim_{\rm FB}$. Secondly, $\approx_{\rm FB}$ and $\approx_{\rm FRB}$ feature the same problem as weak bisimilarity for standard forward-only processes [11], i.e., for $\approx \in \{\approx_{\rm FB}, \approx_{\rm FRB}\}$ it holds that:

$$a \,.\, \underline{0} \;\approx\; \tau \,.\, a \,.\, \underline{0}$$
$$a \,.\, \underline{0} + b \,.\, \underline{0} \;\not\approx\; \tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$$

because if $\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$ performs τ thereby evolving to $\tau^{\dagger} .\, a \,.\, \underline{0} + b \,.\, \underline{0}$ where only a is enabled in the forward direction, then $a \,.\, \underline{0} + b \,.\, \underline{0}$ can neither move nor idle in the attempt to evolve in such a way to match $\tau^{\dagger} .\, a \,.\, \underline{0} + b \,.\, \underline{0}$.

To solve both problems it is sufficient to redefine the two equivalences by making them sensitive to the presence of the past, exactly as in the strong case for forward bisimilarity. By so doing, $a \,.\, \underline{0}$ is no longer identified with $\tau \,.\, a \,.\, \underline{0}$: if the latter performs τ thereby evolving to $\tau^{\dagger} .\, a \,.\, \underline{0}$ and the former idles, then $\tau^{\dagger} .\, a \,.\, \underline{0}$ and $a \,.\, \underline{0}$ are told apart because they are not both initial or non-initial.

Definition 8. *We say that* $P_1, P_2 \in \mathbb{P}$ *are* weakly past-sensitive forward bisimilar*, written* $P_1 \approx_{\mathrm{FB:ps}} P_2$*, iff* $(P_1, P_2) \in \mathcal{B}$ *for some weak past-sensitive forward bisimulation* $\mathcal{B}$*. A symmetric binary relation* $\mathcal{B}$ *over* $\mathbb{P}$ *is a* weak past-sensitive forward bisimulation *iff, whenever* $(P_1, P_2) \in \mathcal{B}$*, then:*

- *initial*$(P_1) \Longleftrightarrow$ *initial*(P_2).
- *Whenever* $P_1 \xrightarrow{\tau} P'_1$*, then* $P_2 \stackrel{\tau^*}{\Longrightarrow} P'_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$.
- *Whenever* $P_1 \xrightarrow{a} P'_1$ *for* $a \in \bar{A}$*, then* $P_2 \stackrel{\tau^*}{\Longrightarrow} \xrightarrow{a} \stackrel{\tau^*}{\Longrightarrow} P'_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$. ∎

Definition 9. *We say that* $P_1, P_2 \in \mathbb{P}$ *are* weakly past-sensitive forward-reverse bisimilar*, written* $P_1 \approx_{\mathrm{FRB:ps}} P_2$*, iff* $(P_1, P_2) \in \mathcal{B}$ *for some weak past-sensitive forward-reverse bisimulation* $\mathcal{B}$*. A symmetric binary relation* $\mathcal{B}$ *over* $\mathbb{P}$ *is a* weak past-sensitive forward-reverse bisimulation *iff, whenever* $(P_1, P_2) \in \mathcal{B}$*, then:*

- *initial*$(P_1) \Longleftrightarrow$ *initial*(P_2).
- *Whenever* $P_1 \xrightarrow{\tau} P'_1$*, then* $P_2 \stackrel{\tau^*}{\Longrightarrow} P'_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$.
- *Whenever* $P_1 \xrightarrow{a} P'_1$ *for* $a \in \bar{A}$*, then* $P_2 \stackrel{\tau^*}{\Longrightarrow} \xrightarrow{a} \stackrel{\tau^*}{\Longrightarrow} P'_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$.
- *Whenever* $P'_1 \xrightarrow{\tau} P_1$*, then* $P'_2 \stackrel{\tau^*}{\Longrightarrow} P_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$.
- *Whenever* $P'_1 \xrightarrow{a} P_1$ *for* $a \in \bar{A}$*, then* $P'_2 \stackrel{\tau^*}{\Longrightarrow} \xrightarrow{a} \stackrel{\tau^*}{\Longrightarrow} P_2$ *and* $(P'_1, P'_2) \in \mathcal{B}$. ∎

Observing that $\sim_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FRB:ps}}$ as the former naturally satisfies the initiality condition, we show the following congruence results. When present, side conditions on subprocesses just ensure that the overall processes are reachable.

Theorem 1. *Let* $\approx \in \{\approx_{\mathrm{FB}}, \approx_{\mathrm{FB:ps}}, \approx_{\mathrm{RB}}, \approx_{\mathrm{FRB}}, \approx_{\mathrm{FRB:ps}}\}$*,* $\approx' \in \{\approx_{\mathrm{FB:ps}}, \approx_{\mathrm{RB}}, \approx_{\mathrm{FRB:ps}}\}$*, and* $P_1, P_2 \in \mathbb{P}$*:*

- *If* $P_1 \approx P_2$ *then for all* $a \in A$*:*
  - $a \,.\, P_1 \approx a \,.\, P_2$ *provided that initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger} .\, P_1 \approx a^{\dagger} .\, P_2$.
- *If* $P_1 \approx' P_2$ *then for all* $P \in \mathbb{P}$*:*
  - $P_1 + P \approx' P_2 + P$ *and* $P + P_1 \approx' P + P_2$ *provided that initial*$(P) \vee$ $($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FB:ps}}$ *is the coarsest congruence with respect to* $+$ *contained in* $\approx_{\mathrm{FB}}$.
- $\approx_{\mathrm{FRB:ps}}$ *is the coarsest congruence with respect to* $+$ *contained in* $\approx_{\mathrm{FRB}}$. ∎

It is worth noting that the aforementioned compositionality problems with respect to alternative composition would not be solved, in this reversible setting, by employing the technique of [11]. If we introduced a variant $\approx'_{\mathrm{FB}}$ of $\approx_{\mathrm{FB}}$ (resp. $\approx'_{\mathrm{FRB}}$ of $\approx_{\mathrm{FRB}}$) such that a τ-transition on either side must be matched by a τ-transition on the other side – possibly preceded and followed by finitely many τ-transitions – with the two reached processes being related by $\approx_{\mathrm{FB}}$ (resp. $\approx_{\mathrm{FRB}}$), then:

- $a^{\dagger} . b . \underline{0} \approx'_{\mathrm{FB}} b . \underline{0}$ but $a^{\dagger} . b . \underline{0} + c . \underline{0} \not\approx'_{\mathrm{FB}} b . \underline{0} + c . \underline{0}$ as explained in Section 2.3.
- Even if $\approx'_{\mathrm{FRB}}$ were a congruence, it would not coincide with $\approx_{\mathrm{FRB:ps}}$ – which is the coarsest congruence inside $\approx_{\mathrm{FRB}}$ – because $a^{\dagger} . (\tau . b . \underline{0} + b . \underline{0}) \approx_{\mathrm{FRB:ps}} a^{\dagger} . b . \underline{0}$ whilst $a^{\dagger} . (\tau . b . \underline{0} + b . \underline{0}) \not\approx'_{\mathrm{FRB}} a^{\dagger} . b . \underline{0}$ as the τ-transition on the left could not be matched by a τ-transition on the right.

5 Equational Characterizations

In this section we investigate the equational characterizations of $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{RB}}$, and $\approx_{\mathrm{FRB:ps}}$ so as to highlight the fundamental laws of these behavioral equivalences. In the following, by deduction system we mean a set comprising the following axioms and inference rules on $\mathbb{P}$ – possibly enriched by a set $\mathcal{A}$ of additional axioms – corresponding to the fact that $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{RB}}$, and $\approx_{\mathrm{FRB:ps}}$ are equivalence relations as well as congruences with respect to action prefix and alternative composition as established by Theorem 1:

- Reflexivity, symmetry, transitivity: $P = P$, $\dfrac{P_1 = P_2}{P_2 = P_1}$, $\dfrac{P_1 = P_2 \quad P_2 = P_3}{P_1 = P_3}$.
- .-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P_1) \wedge \mathit{initial}(P_2)}{a . P_1 = a . P_2}$, $\dfrac{P_1 = P_2}{a^{\dagger} . P_1 = a^{\dagger} . P_2}$.
- +-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P) \vee (\mathit{initial}(P_1) \wedge \mathit{initial}(P_2))}{P_1 + P = P_2 + P \quad P + P_1 = P + P_2}$.

It is known from [2] that, for the three strong bisimilarities, alternative composition turns out to be associative and commutative and to admit $\underline{0}$ as neutral element, like in the case of bisimilarity over standard forward-only processes [7]. The same holds true for $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{RB}}$, and $\approx_{\mathrm{FRB:ps}}$ as they are strictly coarser than their strong counterparts. This is formalized by axioms $\mathcal{A}_1$ to $\mathcal{A}_3$ in Table 2.

Then, we have axioms specific to $\sim_{\mathrm{FB:ps}}$ [2], which are thus valid for $\approx_{\mathrm{FB:ps}}$ too. Axioms $\mathcal{A}_4$ and $\mathcal{A}_5$ together establish that the past can be neglected when moving only forward, but the presence of the past cannot be ignored. Axiom $\mathcal{A}_6$ states that a previously non-selected alternative can be discarded after starting moving only forward.

Likewise, we have axioms specific to $\sim_{\mathrm{RB}}$ [2], which are thus valid for $\approx_{\mathrm{RB}}$ too. Axiom $\mathcal{A}_7$ means that the future can be completely canceled when moving only backward. Axiom $\mathcal{A}_8$ states that a previously non-selected alternative can

$(\mathcal{A}_1)$		$(P_1 + P_2) + P_3$	$= P_1 + (P_2 + P_3)$	
$(\mathcal{A}_2)$		$P_1 + P_2$	$= P_2 + P_1$	
$(\mathcal{A}_3)$		$P + \underline{0}$	$= P$	
$(\mathcal{A}_4)$	$[\sim_{\rm FB:ps}]$	$a^\dagger . P$	$= P$	if $\neg initial(P)$
$(\mathcal{A}_5)$	$[\sim_{\rm FB:ps}]$	$a_1^\dagger . P$	$= a_2^\dagger . P$	if $initial(P)$
$(\mathcal{A}_6)$	$[\sim_{\rm FB:ps}]$	$P + Q$	$= P$	if $\neg initial(P)$, where $initial(Q)$
$(\mathcal{A}_7)$	$[\sim_{\rm RB}]$	$a . P$	$= P$	where $initial(P)$
$(\mathcal{A}_8)$	$[\sim_{\rm RB}]$	$P + Q$	$= P$	if $initial(Q)$
$(\mathcal{A}_9)$	$[\sim_{\rm FB:ps}]$	$P + P$	$= P$	where $initial(P)$
$(\mathcal{A}_{10})$	$[\sim_{\rm FRB}]$	$P + Q$	$= P$	if $initial(Q) \wedge to\_initial(P) = Q$
$(\mathcal{A}_1^\tau)$	$[\approx_{\rm FB:ps}]$	$a . \tau . P$	$= a . P$	where $initial(P)$
$(\mathcal{A}_2^\tau)$	$[\approx_{\rm FB:ps}]$	$P + \tau . P$	$= \tau . P$	where $initial(P)$
$(\mathcal{A}_3^\tau)$	$[\approx_{\rm FB:ps}]$	$a . (P_1 + \tau . P_2) + a . P_2$	$= a . (P_1 + \tau . P_2)$	where $initial(P_1) \wedge initial(P_2)$
$(\mathcal{A}_4^\tau)$	$[\approx_{\rm FB:ps}]$	$a^\dagger . \tau . P$	$= a^\dagger . P$	where $initial(P)$
$(\mathcal{A}_5^\tau)$	$[\approx_{\rm RB}]$	$\tau^\dagger . P$	$= P$	
$(\mathcal{A}_6^\tau)$	$[\approx_{\rm FRB:ps}]$	$a . (\tau . (P_1 + P_2) + P_1)$	$= a . (P_1 + P_2)$	where $initial(P_1) \wedge initial(P_2)$
$(\mathcal{A}_7^\tau)$	$[\approx_{\rm FRB:ps}]$	$a^\dagger . (\tau . (P_1 + P_2) + P_1)$	$= a^\dagger . (P_1 + P_2)$	where $initial(P_1) \wedge initial(P_2)$

Table 2. Axioms characterizing $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, $\approx_{\rm FRB:ps}$

be discarded when moving only backward. Since there are no constraints on P, axiom $\mathcal{A}_8$ subsumes axiom $\mathcal{A}_3$.

Furthermore, the idempotency of alternative composition in the case of bisimilarity over standard forward-only processes, i.e., $P + P = P$ [7], changes as follows depending on the considered equivalence [2]:

- For $\sim_{\rm FB:ps}$, and hence $\approx_{\rm FB:ps}$ too, idempotency is explicitly formalized by axiom $\mathcal{A}_9$, which is disjoint from axiom $\mathcal{A}_6$ where P cannot be initial.
- For $\sim_{\rm RB}$, and hence $\approx_{\rm RB}$ either, an additional axiom is not needed as idempotency follows from axiom $\mathcal{A}_8$ by taking Q equal to P.
- For $\sim_{\rm FRB}$, and hence $\approx_{\rm FRB:ps}$ too, idempotency is formalized by axiom $\mathcal{A}_{10}$, where function *to_initial* brings a process back to its initial version by removing all action decorations:

$$\begin{aligned} to\_initial(\underline{0}) &= \underline{0} \\ to\_initial(a . P) &= a . P \\ to\_initial(a^\dagger . P) &= a . to\_initial(P) \\ to\_initial(P_1 + P_2) &= to\_initial(P_1) + to\_initial(P_2) \end{aligned}$$

  This axiom appeared for the first time in [10] and subsumes axioms $\mathcal{A}_9$ and $\mathcal{A}_6$ for $\sim_{\rm FB:ps}$ and $\approx_{\rm FB:ps}$ as well as axiom $\mathcal{A}_8$ for $\sim_{\rm RB}$ and $\approx_{\rm RB}$.

Let us now focus on axioms specific to $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, and $\approx_{\rm FRB:ps}$, which are usually called τ-laws. Axioms $\mathcal{A}_1^\tau$ to $\mathcal{A}_3^\tau$ are valid for $\approx_{\rm FB:ps}$ and coincide with those for weak bisimulation congruence over standard forward-only processes [7]. A variant of $\mathcal{A}_1^\tau$ with a being decorated, i.e., axiom $\mathcal{A}_4^\tau$, is additionally valid for

$\approx_{\mathrm{FB:ps}}$. As far as $\tau \, . \, P = P$ is concerned, which over standard forward-only processes is valid for weak bisimilarity but not for weak bisimulation congruence [7], its reverse counterpart holds for $\approx_{\mathrm{RB}}$, yielding axiom $\mathcal{A}_5^\tau$. Axioms $\mathcal{A}_6^\tau$ and $\mathcal{A}_7^\tau$ are valid for $\approx_{\mathrm{FRB:ps}}$ and are related to the only τ-law of branching bisimilarity [6].

In the following, we denote by $\vdash$ the deduction relation and we examine the three sets of additional axioms below:

- $\mathcal{A}^\tau_{\mathrm{FB:ps}} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_1^\tau, \mathcal{A}_2^\tau, \mathcal{A}_3^\tau, \mathcal{A}_4^\tau\}$ for $\approx_{\mathrm{FB:ps}}$.
- $\mathcal{A}^\tau_{\mathrm{RB}} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_5^\tau\}$ for $\approx_{\mathrm{RB}}$.
- $\mathcal{A}^\tau_{\mathrm{FRB}} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_{10}, \mathcal{A}_6^\tau, \mathcal{A}_7^\tau\}$ for $\approx_{\mathrm{FRB:ps}}$.

After proving its soundness, we demonstrate the ground completeness of the equational characterization for each of the three considered weak bisimilarities by introducing as usual equivalence-specific normal forms to which every process is shown to be reducible, so that we then work with normal forms only. For each of the three weak bisimilarities, the normal form comes from the one of the corresponding strong bisimilarity in [2] and relies on the fact that alternative composition is associative and commutative, hence the binary $+$ can be generalized to the n-ary $\sum_{i \in I}$ for a finite nonempty index set I.

The proofs of the ground completeness theorems will be done by induction on the *size* of a process, which is inductively defined as follows:

$$\begin{aligned} \mathit{size}(\underline{0}) &= 1 \\ \mathit{size}(a \, . \, P) &= 1 + \mathit{size}(P) \\ \mathit{size}(a^\dagger . \, P) &= 1 + \mathit{size}(P) \\ \mathit{size}(P_1 + P_2) &= \max(\mathit{size}(P_1), \mathit{size}(P_2)) \end{aligned}$$

We also introduce the following function extracting the forward behavior from a process by eliminating executed actions and non-selected alternatives:

$$\begin{aligned} \mathit{to\_forward}(P) &= P && \text{if } \mathit{initial}(P) \\ \mathit{to\_forward}(a^\dagger . \, P) &= \mathit{to\_forward}(P) && \\ \mathit{to\_forward}(P_1 + P_2) &= \mathit{to\_forward}(P_1) && \text{if } \neg \mathit{initial}(P_1) \land \mathit{initial}(P_2) \\ \mathit{to\_forward}(P_1 + P_2) &= \mathit{to\_forward}(P_2) && \text{if } \neg \mathit{initial}(P_2) \land \mathit{initial}(P_1) \end{aligned}$$

which yields an initial process and satisfies the following properties.

Proposition 1. *Let $P, P', P'', Q \in \mathbb{P}$ and $a \in A$:*

- *$\mathit{to\_forward}(P) = P$ iff $\mathit{initial}(P)$, with $\mathit{to\_forward}(P) \approx_{\mathrm{FB}} P$ if $\neg \mathit{initial}(P)$.*
- *$P \xrightarrow{a} P'$ iff $\mathit{to\_forward}(P) \xrightarrow{a} P''$ with $P' \approx_{\mathrm{FB:ps}} P''$.*
- *If $P \approx_{\mathrm{FB:ps}} Q$, then $\mathit{to\_forward}(P) \approx_{\mathrm{FB:ps}} \mathit{to\_forward}(Q)$ when P and Q are initial or cannot execute τ-actions, else $\mathit{to\_forward}(P) \approx_{\mathrm{FB}} \mathit{to\_forward}(Q)$.* ■

We start by showing the soundness and ground completeness of $\mathcal{A}^\tau_{\mathrm{FB:ps}}$ with respect to $\approx_{\mathrm{FB:ps}}$.

Theorem 2. *Let $P_1, P_2 \in \mathbb{P}$. If $\mathcal{A}^\tau_{\mathrm{FB:ps}} \vdash P_1 = P_2$ then $P_1 \approx_{\mathrm{FB:ps}} P_2$.* ■

Definition 10. *We say that $P \in \mathbb{P}$ is in $\approx_{\mathrm{FB:ps}}$-*normal form*, written $\approx_{\mathrm{FB:ps}}$-nf, iff it is equal to one of the following:*

– $\underline{0}$.
– $\sum_{i \in I} a_i \, . \, P_i$, *where each* P_i *is initial and in* $\approx_{\mathrm{FB:ps}}$*-nf.*
– $a^{\dagger} . \, P'$, *where* P' *is initial and in* $\approx_{\mathrm{FB:ps}}$*-nf.* ■

Lemma 1. *For all* $P \in \mathbb{P}$ *there is* $Q \in \mathbb{P}$ *in* $\approx_{\mathrm{FB:ps}}$*-nf such that* $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$. ■

Following the approach adopted in [11] for weak bisimulation congruence over standard forward-only processes, for $\approx_{\mathrm{FB:ps}}$ we introduce a further normal form where, unlike [11], two distinct equivalent processes P' and P'' come into play instead of a single process due to the presence of action decorations within processes in our reversible setting. This leads to the so-called *saturation lemma*, which immediately follows the definition below and, unlike [11], features *to_forward*(P') in place of P' in the final part of its statement.

Definition 11. *We say that* $P \in \mathbb{P}$ *is in* $\approx_{\mathrm{FB:ps}}$-full normal form*, written* $\approx_{\mathrm{FB:ps}}$*-fnf, iff it is equal to one of the following:*

– $\underline{0}$
– $\sum_{i \in I} a_i \, . \, P_i$, *where each* P_i *is initial and in* $\approx_{\mathrm{FB:ps}}$*-fnf*
– $a^{\dagger} . \, P'$, *where* P' *is initial and in* $\approx_{\mathrm{FB:ps}}$*-fnf*

and whenever $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$, *then* $P \stackrel{a}{\longrightarrow} P''$ *with* $P' \approx_{\mathrm{FB:ps}} P''$. ■

Lemma 2. *Let* $P \in \mathbb{P}$ *be initial. If* $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ *then* $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \, . \, \mathit{to\_forward}(P')$. ■

Lemma 3. *For all* $P \in \mathbb{P}$ *in* $\approx_{\mathrm{FB:ps}}$*-nf there is* $Q \in \mathbb{P}$ *in* $\approx_{\mathrm{FB:ps}}$*-fnf such that* $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$. ■

Theorem 3. *Let* $P_1, P_2 \in \mathbb{P}$. *If* $P_1 \approx_{\mathrm{FB:ps}} P_2$ *then* $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_1 = P_2$. ■

As for the soundness and ground completeness of $\mathcal{A}^{\tau}_{\mathrm{RB}}$ with respect to $\approx_{\mathrm{RB}}$, the latter does not require saturation as no choice occurs when going backward.

Theorem 4. *Let* $P_1, P_2 \in \mathbb{P}$. *If* $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = P_2$ *then* $P_1 \approx_{\mathrm{RB}} P_2$. ■

Definition 12. *We say that* $P \in \mathbb{P}$ *is in* $\approx_{\mathrm{RB}}$-normal form*, written* $\approx_{\mathrm{RB}}$*-nf, iff it is equal to one of the following:*

– $\underline{0}$.
– $a^{\dagger} . \, P$, *where* P *is in* $\approx_{\mathrm{RB}}$*-nf.* ■

Lemma 4. *For all* $P \in \mathbb{P}$ *there is* $Q \in \mathbb{P}$ *in* $\approx_{\mathrm{RB}}$*-nf such that* $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P = Q$. ■

Theorem 5. *Let* $P_1, P_2 \in \mathbb{P}$. *If* $P_1 \approx_{\mathrm{RB}} P_2$ *then* $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = P_2$. ■

We conclude by showing the soundness of $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}}$ with respect to $\approx_{\mathrm{FRB:ps}}$.

Theorem 6. *Let* $P_1, P_2 \in \mathbb{P}$. *If* $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$ *then* $P_1 \approx_{\mathrm{FRB:ps}} P_2$. ■

The investigation of ground completeness, which we conjecture to hold on the basis of the fact that weak back-and-forth bisimilarity coincides with branching bisimilarity [4], is left for future work. We recall from [5] that saturation is unsound for branching bisimulation semantics over standard forward-only processes, hence a different proof technique is necessary.

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