Tropical Mathematics and the Lambda-Calculus

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Abstract. We study the interpretation of the lambda-calculus in a framework based on tropical mathematics, by considering the relational semantics weighted on the tropical semiring. We show it provides a framework where both program metrics, based on the analysis of program sensitivity via Lipschitz conditions, and resource analysis, based on higher-order program differentiation, coexist.

1 Introduction

In recent years, more and more interest in the programming language community has been directed towards the study of quantitative properties of programs like computing the number of computation steps or convergence probabilities, as opposed to purely qualitative properties like termination or program equivalence. In particular, two different quantitative approaches have received considerable attention: On the one hand, there is the approach of *program metrics* [2,3,30] and *quantitative equational theories* [27], based on the observation that probabilistic or numerical algorithms are not thought to compute a target function f *exactly*, but only in an approximate way. This led to study denotational frameworks in which types are endowed with metrics measuring similarities in program behavior [30], [4], [10,16,29]. On the other hand, there is the approach based on *differential* [14], [14], [1,8,22] or *resource-aware* [7] extensions of the λ-calculus, which is well-connected to the *relational semantics* [13,22,26] and *non-idempotent* intersection types [11, 28]. This led to study syntactic or denotational frameworks in which one can define a *Taylor expansion* of programs.

In both approaches a crucial role is played by the notion of *linearity*, in the sense of linear logic, i.e. of using inputs exactly once. In metric semantics, linear programs correspond to *non-expansive* functions, i.e. maps that do not increase distances; moreover, the possibility of duplicating inputs leads to interpret programs with a fixed duplication bound as *Lipschitz-continuous* maps [2]. By contrast, in the standard semantics of the differential λ-calculus, linear programs correspond to linear maps, in the usual algebraic sense, while the possibility of duplicating inputs gives rise to *power series*.

The starting observation of this work is that, at a first glance, there seems to be a "logarithmic" gap between the two approaches: in metric models n times duplication results in a n-Lipschitz *linear* function $n \cdot x$, while in differential models this results in a non-Lipschitz *polynomial* function x^n. At the same time, this gap may be overcome once we interpret these functions in the framework of tropical mathematics where, e.g., x^n precisely reads as $n \cdot x$. Tropical mathematics [31] is a well established algebraic and geometrical framework, with tight connections with optimisation theory [25], where the usual ring structure of numbers based on addition and multiplication is replaced by the semiring structure given, respectively, by "min" and "+". For instance, the polynomial $p(x, y) = x^2 + xy^2 + y^3$, when interpreted over the tropical semiring, translates as the piecewise linear function $\mathtt{t}f(x, y) = \min\{2x, x + 2y, 3y\}$.

A tropical variant of denotational semantics has already been considered [22], and shown capable of capturing *best-case* quantitative properties. Connections between tropical linear algebra and metric spaces have also been observed [15] within the abstract setting of *quantale-enriched* categories [19, 32]. However, a thorough investigation of the full power of the interpretation of the λ-calculus within tropical mathematics has not yet been undertaken. We sketch here some first steps. The aim is to bridge the two approaches mentioned above by making them *coexist*, and suggesting the application of tropical methods to the study of the λ-calculus and its quantitative extensions. For instance, we could show that "tropical interpretations" are related to likelihoods functions of the different reduction paths of probabilistic calculi, or that they scale to a more abstract level, leading to introduce a differential operator for continuous functors between *generalized* metric spaces ([23]). However, we will not discuss these points here. Instead we will focus on, first, recall the metric and differential approaches to linearity, second, sketching how the tropical semantics makes them coexist.

2 Resource control in λ-calculi

Graded typed λ-calculus The language bSTLC we consider is a somehow simplified version of the language Fuzz [30], or of graded linear logic [17].

Terms are as for the simply typed λ-calculus (STLC), types are $A ::= X \mid\ !_n A \multimap A$, judgements' contexts are declarations $x :_n A$ and typing rules are:

$$\frac{\Gamma \vdash M : A}{\Gamma, x :_0 B \vdash M : A} \qquad \frac{\Gamma, x :_n B, y :_m B \vdash M : A}{\Gamma, x :_{n+m} B \vdash M[x/y] : A} \qquad \frac{\Gamma, x :_n A \vdash M : B}{\Gamma \vdash \lambda x.M :!_n A \multimap B} \qquad \frac{\Gamma \vdash M :!_n A \multimap B \quad \Delta \vdash N : A}{\Gamma + n\Delta \vdash MN : B}$$

where $n \in \mathbb{N}$, $\Gamma + \Delta$ is defined by $(\Gamma, x :_m A) + (\Delta, x :_n A) = (\Gamma + \Delta), x :_{m+n} A$, and $m\Gamma$ is made all $x :_{mn} A$ for $(x :_n A) \in \Gamma$. The axiom is $x :_1 A \vdash x : A$. The main feature of this language is that if $\vdash \lambda x.M :!_n A \multimap B$, then x will be duplicated exactly n times in the reduction to the normal form. E.g., $\vdash_{\mathsf{bSTLC}}$ $\lambda z.\, (\lambda xy.yxx)\, z :!_2 X \multimap !_1 (!_1 X \multimap !_1 X \multimap X) \multimap X$, where the colored grading indicate the number of time the respective terms will be duplicated. The bSTLC can be interpreted in a symmetric monoidal closed category (SMCC) equipped with a $\mathbb{N}$-*graded linear exponential comonad* [20].

Differential λ-calculus The differential λ-calculus ST∂LC (e.g. [9, Section 3]), is given by *terms* M and *sums* $\mathbb{T}$, mutually generated by: $M ::= x \mid \lambda x.M \mid M\mathbb{T} \mid \mathsf{D}[M] \cdot M$ and $\mathbb{T} ::= 0 \mid M \mid M + \mathbb{T}$, quotiented by a number of equations (e.g. α-equivalence, or linearity of some constructors). We follow the tradition of quotienting also for the idempotency of $+$. Simple types are $A ::= X \mid A \to A$. The axioms are $\Gamma, x : A \vdash x : A$ and $\Gamma \vdash 0 : A$. The typing rules:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash \mathbb{T} : A}{\Gamma \vdash M\mathbb{T} : B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash \mathsf{D}[M] \cdot N : A \to B} \qquad \frac{\Gamma \vdash M_1 : A \cdots \Gamma \vdash M_n : A}{\Gamma \vdash M_1 + \cdots + M_n : A}\,(n \geq 2)$$

Linearity is handled via the operational semantics (that we do not give) ensuring exact duplication control: e.g., writing $\mathsf{D}^2[\_] \cdot (\_)^2$ as a shortcut for $\mathsf{D}[\mathsf{D}[\_] \cdot (\_)] \cdot (\_)$, the analogue of the previous bSTLC-term is $\vdash_{\mathrm{ST}\partial\mathrm{LC}}$ $\lambda z. \big(\mathsf{D}^2[\lambda xy. \big(\mathsf{D}^1[\big(\mathsf{D}^1[y] \cdot x^1\big)\, 0] \cdot x^1\big)\, 0] \cdot z^2\big)\, 0 \,:\, X \to (X \to X \to X) \to X$. In particular, if the *multiplicities* of the arguments (the colored exponents) do not exactly match the number of duplications, the term reduces to the empty "error" sum 0. The ST∂LC can be interpreted in Cartesian closed differential λ-categories (CC∂λC) [5, 6, 9]. In them, homsets are equipped with a structure of commutative monoid and with a *differential operator* D. E.g. the *CC∂λ*C of convenient vector spaces with smooth maps, where D is the usual differential of smooth maps. Finally, in ST∂LC we can perform a syntactic Taylor expansion of an ordinary λ-term via an inductively defined map $\mathcal{T}()$ giving rise to an infinite series of terms: $\mathcal{T}(MN) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \left(\mathsf{D}^n[\mathcal{T}(M)] \cdot \mathcal{T}(N)^n\right) 0$. As in analysis, it decomposes an application as a series of k-linear functions, which can be seen as its approximants. Since we consider idempotent sum, the factorial coefficients disappear and the resulting map is called the *qualitative* Taylor expansion.

3 Tropical Weighted relational semantics

Tropical mathematics in a nutshell We let the *tropical semiring* $\mathbb{L}$ be $[0, \infty]$ with addition min and multiplication $+$ or, equivalently, the *Lawvere quantale* [19, 32] $[0, \infty]$ with order $\geq$ and usual $+$ as monoid action. $\mathbb{L}$ is at the heart of both tropical mathematics and the categorical study of metric spaces [23]. A *tropical polynomial* is a piecewise linear function $\varphi : \mathbb{L} \to \mathbb{L}$ of shape $\varphi(x) = \min_{j=1,\dots,k}\{i_j x + \widehat{\varphi}_{i_j}\}$, with $i_j \in \mathbb{N}$, $\widehat{\varphi}_{i_j} \in \mathbb{L}$. Those are always Lipschitz functions. E.g., $\varphi_n(x) = \min_{i \leq n}\{ix + 2^{-i}\}$, in Fig 1. A *tropical power series* (of one variable), shortly a *tps*, is a function $\varphi : \mathbb{L} \to \mathbb{L}$ of the shape $\varphi(x) = \inf_{n \in \mathbb{N}}\{nx + \widehat{\varphi}_n\}$, with $\widehat{\varphi}_n \in \mathbb{L}$. This is a "limit" of tropical polynomials of higher and higher degree. E.g., $\varphi(x) := \inf_{i \in \mathbb{N}}\{ix + 2^{-i}\}$ is the "limit" of the φ_n, see Fig 1. Tps are in general not Lipschitz, and their study is less developed than that of tropical polynomials.

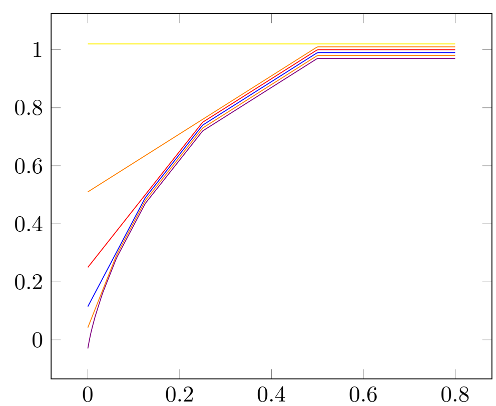

Fig. 1: Tropical polynomials $\varphi_0, \dots, \varphi_4$ (top to bottom), and their limit tps φ (in violet).

Tropical weighted relational semantics in a nutshell A well-known of the λ-calculus and linear logic is the *weighted relational semantics*: for a fixed *continuous* semi-ring Q, the category $Q\mathsf{Rel}$ has objects the sets and $Q\mathsf{Rel}(X,Y) = Q^{X\times Y}$ (set-indexed matrices with coefficients in Q). As expected, Q^X is a Q-module and we can identify $Q\mathsf{Rel}(X,Y)$ with the set of linear maps from Q^X to Q^Y. Taking $Q := \mathbb{L}$ we obtain the *tropical weighted relational model* $\mathbb{L}\mathsf{Rel}$. Remark that the composition in $\mathbb{L}\mathsf{Rel}$ reads as $(s\circ t)_{a,c} := \inf_{b\in Y}\{s_{b,c}+t_{a,b}\}$. By applying known results (e.g. [22], [21], [24]), one sees that $\mathbb{L}\mathsf{Rel}$ gives rise to denotational models of several variants of the STLC: If we let $!_n X$:= finite multisets on X with cardinality $\leq n$, then $(!_n)_{n\in\mathbb{N}}$ lifts to a $\mathbb{N}$-graded linear exponential comonad over $\mathbb{L}\mathsf{Rel}$, i.e. gives a model of bSTLC; If we let $!X$:= finite multisets on X, the coKleisli $\mathbb{L}\mathsf{Rel}_!$ is CCC, i.e. a model of STLC. If we let a *differential operator* $D : \mathbb{L}\mathsf{Rel}(!X,Y) \to \mathbb{L}\mathsf{Rel}(!(X+X),Y)$ be given by: $(Dt)_{\mu\oplus\rho,b} := t_{\rho+\mu,b}$ if $\#\mu = 1$ and $:= \infty$ otherwise, then $\mathbb{L}\mathsf{Rel}_!$ becomes a CC$\partial\lambda$C, i.e. a model of ST∂LC. Moreover, it can be seen that the morphisms of $\mathbb{L}\mathsf{Rel}_!$ can always be Taylor expanded (for a suited notion of Taylor expansion in a CC$\partial\lambda$C), and the series interpreting in $\mathbb{L}\mathsf{Rel}_!$ the Taylor expansion of a STLC-term M, converges to the interpretation of M. Finally notice that, as usual, since a morphism $t\in\mathbb{L}\mathsf{Rel}_!(X,Y)$ is a matrix, it yields a *linear* map $\mathbb{L}^{!X}\to\mathbb{L}^Y$. But we can also "express it in the base X" and see it as a *non-linear* map $t^! : \mathbb{L}^X \to \mathbb{L}^Y$.This is made possible by the coKleisli structure, and concretely one founds $t^!(x)_b := \inf_{\mu\in !X}\{\mu x + t_{\mu,b}\}$ where $\mu x := \sum_{a\in X}\mu(a)x_a$. These functions correspond then to generalised tps (possibly infinitely many variables), and for $X = Y = \{*\}$, we get usual tps of one variables. Instead, if $\{\mu \in !X \mid \widehat{f}_{\mu,b} \neq \infty\}$ is *finite*, we get tropical polynomials in possibly infinitely many variables.

4 Tropical Metric Semantics

The main goal of this section is to show that the interpretation of the above mentioned variants of the STLC based on $\mathbb{L}\mathsf{Rel}$ yield a metric semantics, where the spaces $\mathbb{L}^X$ are endowed with the $\|\cdot\|_\infty$-norm metric:

Theorem 1. *For any λ-term M:*

1. *if $\Gamma \vdash_{\mathsf{b}\mathrm{STLC}} M : A$, then $[\![M]\!]^! : \mathbb{L}^{[\![\Gamma]\!]} \to \mathbb{L}^{[\![A]\!]}$ is a* Lipschitz *map.*
2. *if $\Gamma \vdash_{\mathrm{STLC}} M : A$, then $[\![M]\!]^! : \mathbb{L}^{[\![\Gamma]\!]} \to \mathbb{L}^{[\![A]\!]}$ is a* locally Lipschitz *map. Moreover, the Taylor expansion $\mathcal{T}(M)$ decomposes $[\![M]\!]^!$ into an* inf *of* Lipschitz *maps.*

Recall that the syntactic Taylor expansion decomposes an unbounded application as a limit of bounded ones; the result above lifts this decomposition to a semantic level, presenting a higher-order program as limits of Lipschitz maps: it provides thus a bridge between the metric and the differential approaches. Its proof requires the study of tps with mathematical analysis tools, sketched below.

It is not hard to see that any tps $f : \mathbb{L}^X \to \mathbb{L}^Y$ is non-decreasing and concave, w.r.t. the pointwise order, and continuous. Moreover, tropical *linear* functions

$f : \mathbb{L}^X \to \mathbb{L}^Y$ are non-expansive and, more generally if a tps $f : \mathbb{L}^X \to \mathbb{L}^Y$ arises from a matrix $\widehat{f} : !_n X \times Y \to \mathbb{L}$, then f is a n-Lipschitz map. This is in perfect analogy with what happens in the metric models recalled in the introduction.

Consider now the case of tps with *finitely many variables*, e.g. the one shown in Fig 1: the tps $\varphi(x) = \inf_i\{x + 2^{-i}\}$ behaves *locally* like the polynomials $\varphi_n(x) = \min_{i \leq n}\{x+2^{-i}\}$. However, at $x = 0$ we have that $\varphi(0) = \inf_{i \in \mathbb{N}} 2^{-i} = 0$, and this is the only point where the inf is *not* a min. Also, while the derivative of φ is bounded on all $\mathbb{R}_{>0}$, at $x = 0^+$ it tends to ∞. This is reminiscent of [12, Example 7]. These properties are shared by all tps with finitely many variables, as Theorem 2 shows (identify $!\{1, \dots, k\}$ with $\mathbb{N}^k$, so the matrix of a tps f with variables $x = x_1, \dots, x_k$ is a $\widehat{f} : \mathbb{N}^k \to \mathbb{L}$, and $f(x) = \inf_{n \in \mathbb{N}^k}\{nx + \widehat{f}(n)\}$, with nx the scalar product).

Theorem 2. *Let $k \in \mathbb{N}$ and $f : \mathbb{L}^k \to \mathbb{L}$ a tps with matrix $\widehat{f} : \mathbb{N}^k \to \mathbb{L}$. For all $0 < \epsilon < \infty$, there is a* finite *$\mathcal{F}_\epsilon \subseteq \mathbb{N}^k$ such that $f\big|[\epsilon, \infty]^k$ coincides with the tropical* polynomial *$P_\epsilon(x) := \min_{n \in \mathcal{F}_\epsilon}\{nx + \widehat{f}(n)\}$.*

The result above suggests that, far from 0, tps with finitely many variables can be studied with the tools of tropical geometry (e.g. tropical roots, Newton polygones). The question is thus what do these tools tell about λ-terms. Moreover, a consequence of Theorem 2 is that all tps with finitely many variables are always *locally* Lipschitz on $\mathbb{R}_{>0}$. By some convex analysis argument we finally have:

Theorem 3. *All tps $\mathbb{L}^X \to \mathbb{L}$ are locally Lipschitz on $\mathbb{R}^X_{>0}$.*

Let us conclude by mentioning that the differential operator D of $\mathbb{L}\mathsf{Rel}_!$ translates into a differential operator $D_!$ sending a tps $f : \mathbb{L}^X \to \mathbb{L}^Y$ to a tps $D_! f : \mathbb{L}^X \times \mathbb{L}^X \to \mathbb{L}^Y$, linear in its first variable, $D_! f(x,y)_b = \inf_{a \in X, \mu \in !X}\left\{\widehat{f}_{\mu+a} + x_a + \mu y\right\}$. When f is a tropical polynomial, $D_! f$ actually relates to the standard tropical derivative [18]. The Taylor formula for $\mathbb{L}\mathsf{Rel}_!$ morphisms becomes a "tropical" Taylor formula for tps: $f(x) = \inf_n\left\{D_!^{(n)}(f)(!_n x, \infty)\right\}$.

5 Conclusion

After recalling the metric and the differential approach to linearity in the λ-calculus, the main goal of this short contribution is to demonstrate the existence of a conceptual bridge between such two well-studied quantitative approaches to higher-order programs, and to highlight the possibility of transferring results and techniques from one approach to the other. For instance, Theorem 1 allows metric considerations on the Taylor expansion of programs. We think that topical mathematics, a field which has been largely and successfully applied in computer science, could be also used to study quantitative properties of higher-order programs. In fact, the real interest of the weighted relational semantics is the interpretation of effectful programs, which we did not consider here, but which is indeed possible and we are currently investigating, e.g., its relations with optimisation problems of probabilistic calculi.

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