$L(3,2,1)$-Labeling of Certain Planar Graphs

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Abstract. Given a graph $G = (V, E)$ of maximum degree Δ, denoting by $d(x, y)$ the distance in G between nodes $x, y \in V$. An $L(3, 2, 1)$-labeling of G is an assignment l from V to the set of non-negative integers such that $|l(x) - l(y)| \geq 3$ if x and y are adjacent, $|l(x) - l(y)| \geq 2$ if $d(x, y) = 2$, and $|l(x) - l(y)| \geq 1$ if $d(x, y) = 3$, for all x and y in V. The $L(3, 2, 1)$-number $\lambda(G)$ is the smallest positive integer such that G admits an $L(3, 2, 1)$-labeling with labels from $\{0, 1, \ldots, \lambda(G)\}$.
In this paper, the $L(3, 2, 1)$-number of certain planar graphs is determined, proving that it is linear in Δ, although the general upper bound for the $L(3, 2, 1)$-number of planar graphs is quadratic in Δ.

Keywords: $L(h, k)$-labeling · frequency assignment problems · infinite grids · square of cycles · outerplanar graphs.

1 Introduction

Given a graph $G = (V, E)$, denote by Δ its maximum degree, and let $d(x, y)$ be the distance in G between nodes $x, y \in V$. An $L(3, 2, 1)$-labeling of G is an assignment l from V to the set of non-negative integers such that $|l(x) - l(y)| \geq 3$ if x and y are adjacent, $|l(x) - l(y)| \geq 2$ if $d(x, y) = 2$, and $|l(x) - l(y)| \geq 1$ if $d(x, y) = 3$, for all x and y in V. The $L(3, 2, 1)$-number $\lambda(G)$ is the smallest positive integer such that G admits an $L(3, 2, 1)$-labeling with labels from $\{0, 1, \ldots, \lambda(G)\}$.

In the context of frequency assignment problems in ad-hoc wireless networks, transmitters are assigned frequencies so that they are at a mutual distance at least equal to a minimum allowed separation, and the aim is to minimize the used bandwidth. Hale [14] introduced a graph model for this problem already in 1980. Roberts [19] introduced the concept of 'very close' and 'close' stations (respectively at a distance 1 and 2 in the corresponding communication graph) and, in 1992, Griggs and Yeh [13] formulated the problem in terms of a special graph coloring problem called $L(2, 1)$-labeling problem. The $L(2, 1)$-labeling problem and its more general version, the $L(h, k)$-labeling problem, have been the subject of a huge number of papers, most of them devoted to proving the

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very well known Griggs and Yeh's conjecture, concerning the maximum possible value of the bandwidth (for a survey, see [5]).

In practice, interference among frequencies could go beyond distance two so, in 2004, Liu and Shao [17] generalized the $L(2,1)$-labeling problem to the $L(3,2,1)$-labeling problem to take into account even stations at a distance 3.

Since its definition, the technology has changed, and the $L(3,2,1)$-labeling problem became outdated in practice; nevertheless, it has been considered attractive by researchers even only from a purely theoretical point of view, and many papers have been published on this topic, in an attempt to clarify which is the maximum necessary bandwidth. This paper goes in this direction.

The problem of deciding whether $\lambda(G)$ is upper bounded by a given parameter k is trivially NP-complete because, for example, it coincides with the decisional version of the $L(2,1)$-labeling problem on diameter 2 graphs, which is difficult [13].

In general, if G is a graph with maximum degree Δ, Clipperton et al. [8] proved that $\lambda(G) \leq \Delta^3 + \Delta^2 + 3\Delta$; later, this upper bound was improved to $\Delta^3 + 2\Delta$ [7].

The $L(3,2,1)$-number of many graphs is known; for example, paths, cycles, caterpillars, complete and complete bipartite graphs [8]; fans and wheels [18]; interval [1], permutation [2] and trapezoid graphs [3]; power of paths [7]; the cartesian product of paths and cycles [7], of a complete bipartite graph and a path or a cycle [12], and of a triangle and a cycle in [16].

Liu and Shao [17] showed that $\lambda(G) \leq 15(\Delta^2 - \Delta + 1)$ if G is a planar graph of maximum degree Δ.

Nevertheless, for interesting subclasses of planar graphs, better results are known:

- n-length paths P_n and cycles C_n, $n \geq 8$: $\lambda(P_n) = 7$, $\lambda(C_n) = 7$ if n is even and $= 8$ if n odd (but also the results for small values of n are found) [8];
- ladders $L_n = P_n \times P_2$: $\lambda(L_n) = 9$ if $n \geq 5$ (but also the results for small values of n are found) [7];
- caterpillars C: $\lambda(C) \leq 2\Delta + 2$ [8];
- trees T: $2\Delta + 1 \leq \lambda(T) \leq 2\Delta + 3$, deciding which is the exact value for the $L(3,2,1)$-number is NP-complete in general, while the upper bound is tight for complete $(\Delta - 1)$-ary trees and the lower bound is tight for stars with n+1 nodes [7];
- wheels W_n with $n + 1$ nodes (and of degree $\Delta = n$): $\lambda(W_n) = 2\Delta + 1$ [18]; particular star- and wheel-related graphs that turn out to be planar are studied in [11] and their $L(3,2,1)$-number is also linear in Δ;
- friendship graphs $Fr_n = nK_2 + K_1$: $\lambda(Fr_n) = 4n + 1 = 2\Delta + 1$ [18].

It is evident that, despite the general quadratic upper bound, for some classes of planar graphs, the upper bound on $\lambda(G)$ is linear in Δ. We will prove this is true also for other important subclasses of planar graphs, *i.e.*, regular grids, the square of cycles, and outerplanar graphs.

2 Definitions and Preliminary results

Definition 1. *Let $G = (V, E)$ be a graph and l be a mapping $l : V \to \mathbb{N} \cup \{0\}$. l is an* $L(3, 2, 1)$-labeling *of G if, for all $x, y \in V$,*

$$|l(x) - l(y)| \geq \begin{cases} 3, if d(x,y) = 1 \\ 2, if d(x,y) = 2 \\ 1, if d(x,y) = 3 \end{cases} \tag{1}$$

Definition 2. *Given a graph $G = (V, E)$ and an $L(3, 2, 1)$-labeling l of it, the value $\sigma(l, G) = \max_{v \in V} l(v)$ is called* span *of l. The minimum value of $\sigma(l, G)$ over all mapping functions l for G is called the* $L(3, 2, 1)$-number of *G and denoted by $\lambda(G)$ (simply λ for short, where no confusion arises).*

It is not restrictive to assume that, given a graph G, our labeling l is such that $l(v) = 0$ for some node v of G, because otherwise it is possible to obtain a new labeling with this property by shifting the values of all the labels from $l(v)$ to $l(v) - \min_{u \in V} l(u)$ for each $v \in V$.

An $L(3, 2, 1)$-labelling l of a graph G with span $\sigma(l, G)$ is *cyclic* if the difference $|l(x) - l(y)|_c$ between two labels $l(x)$ and $l(y)$ is defined as $|l(x) - l(y)|_c = \min\{|l(x) - l(y)|, \sigma(l, G) + 1 - |l(x) - l(y)|\}$. In other words, in cyclic labelling, the first and last labels are considered at distance 1. The *cyclic $L(3, 2, 1)$-labeling number* of G is denoted by $\lambda^c(G)$ (simply λ^c for short, where no confusion arises).

This concept has been introduced in [15] concerning another labeling function, the $L(2, 1)$-labeling.

Straightforwardly, a feasible cyclic $L(3, 2, 1)$-labeling is always a feasible $L(3, 2, 1)$-labeling, while the vice-versa is not necessarily true, so, for any graph G, $\lambda^c(G) \geq \lambda(G)$.

Given a graph G, the values of $l(v)$, for any $v \in V$, are interchangeably called *labels* and *colors*. Because of this name, it is common to define *palette* of an $L(3, 2, 1)$-labeling function l of G the set $P = \{0, 1, \ldots, \sigma(l, G)\}$.

For each graph G, it holds the *symmetry property* of the palette: it is always possible to get a new labeling l' starting from a given one l by simply assigning $l'(v) = \sigma(l, G) - l(v)$; trivially, $\sigma(l, G) = \sigma(l', G)$.

Given an $L(3, 2, 1)$-labeling function l for G, if $l(v)$ has already been determined for some v, then we will choose the colors to label the nodes at a distance ≤ 3 from v in G among the ones in a subset obtained from the palette by *temporarily eliminating* $l(v)$ and possibly other close colors. Some proofs will exploit this concept of deleting some colors from the palette for a certain time.

In this paper, we consider the following subclasses of planar graphs.

The *infinite hexagonal grid* is the graph naturally derived from a tassellation of the plane with regular hexagons. Since it is a regular graph of degree 3, we will name it G_3. For the sake of completeness, we analogously define the *infinite squared and triangular grids* as derived from a tassellation of the plane with squares and equilateral triangles, respectively, which are regular graphs of degree 4 and 6, and hence denoted G_4 and G_6. Finally, it is also possible to generalize the concept of regular grid to not planar graphs and define the *infinite octagonal grid*, which does not come from any tassellation but is a regular graph of degree 8, call it G_8.

A *cycle* is an ordered sequence of nodes $v_1, \ldots, v_n$ connected by edges $\{v_i, v_{i+1}\}$, $i = 1, \ldots, n-1$, and $\{v_n, v_1\}$. C_n denotes an n node cycle.

The *square* of a graph G is a graph G^2 that has the same set of nodes as G, and two nods are adjacent when their distance in G is at most 2. We focus on the square of cycles C_n^2, which are planar graphs when n is even and when $n = 3$.

A graph is *outerplanar* if it can be embedded in the plane so that every node lies on the boundary of the outer face. It follows that once the first node has been chosen, clockwise order around this face induces a total order on the graph nodes.

We now present some preliminary results that will be exploited in the following.

Theorem 1. *For any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \geq 2\Delta + 1$.*

Proof. This lower bound trivially comes from observing that the Δ neighbors of a maximum degree node v must be labeled with Δ labels at mutual distance 2 (so at least $\Delta + \Delta - 1$ colors) and that v must receive a different label at a distance 3 from any one of its neighbors. □

A *fan* F_n has $n+1$ nodes; n of them, called $v_1, \ldots, v_n$, constitute a path, while one, called c, is connected by an edge to all the other n.

Murugan and Surija [18] proved that, for $n \geq 5$, $\lambda(F_n) = 2n+1$. Here we complete this result also for small values of n:

Lemma 1. $\lambda(F_n) = 2n + 1$ *if* $n \geq 4$ *while* $\lambda(F_n) = 2n + 2$ *if* $n = 2, 3$.

Proof. First, observe that a possible general labeling of F_n assigns to c label 0, and ordinately from left to right the following sequence of labels to $v_1, \ldots, v_n$: it starts with color 3 and hops colors 4 by 4; when the end of the palette is reached, it begins again from color 5 and hops again 4 colors by 4. (For example, if $n = 5$, $v_1, \ldots, v_5$ will be labeled by labels 3, 7, 11, 5, 9 in this order.)

If $n = 2$, F_n is a triangle, and $\lambda(F_2) = 2n + 2 = 6$. All three nodes must get labels at mutual distance 3, *i.e.*, 0, 3, and 6.

If $n = 3$, we prove that $\lambda(F_3) = 8$.

First we prove that $\lambda(F_3) > 7$. By contradiction, assume that the palette $P = \{0, 1, \ldots, 7\}$. If $l(c) = 0$, then v_1, v_2 and v_3 must necessarily be labeled with 3, 5, 7 in some order; but no one among these three colors is suitable to be the label of v_2, that is adjacent both to v_1 and to v_3. If $l(c) = 7$, similar reasonings hold. If finally, c is labeled with a color different both from 0 and from 7, it is easy to see that there are not three available labels at mutual distance 2.

On the other hand, $\lambda(F_3) \leq 8$ because a possible feasible labeling of this graph assigns 0 to node c and the labels 6, 3, 8 to v_1, v_2, v_3.

So, $\lambda(F_3) = 2n + 2 = 8$.

If $n = 4$, we apply Theorem 1, deducing that $\lambda(F_4) \geq 9$. A possible labeling assigns 0 to node c and the sequence 7, 3, 9, 5 to $v_1, \ldots, v_4$ in this order. It follows that $\lambda(F_4) = 2n + 1 = 9$. (Note that the labeling procedure used for $n \geq 5$ would produce for $v_1, \ldots, v_4$ the sequence 3, 7, 5, 9 that is not feasible because colors 5 and 7 are assigned to adjacent nodes.) □

Since we want to exploit a feasible $L(3,2,1)$-labeling of a fan as a building block for more general results, we now assume that c is pre-colored and provide a labeling algorithm in the proof of the following result.

Theorem 2. *Let be given a fan F_n whose node c has been pre-colored with label $l(c)$. Then, the $L(3,2,1)$-labeling l of F_n can be completed with span $\sigma(l, F_n) \leq 2n + 3$ if $n \geq 4$ while $\sigma(l, F_n) \leq 2n + 4$ if $n = 2, 3$. If $l(c) \neq \{0, 1, \sigma(l, F_n) - 1, \sigma(l, F_n)\}$ then $\lambda(F_n) = 2n + 3$ if $n \geq 4$ and $\lambda(F_n) = 2n + 4$ if $n = 2, 3$.*

Proof. We provide an $L(3,2,1)$-labeling procedure for nodes $v_1, \ldots, v_n$; the resulting span is trivially obtained by considering the largest used label.

Assume $n \geq 5$, first. In the reasonings under the proof of the previous lemma, colors 3, 4, $\ldots, 2n + 1$ were available and sufficient for labeling $v_1, \ldots, v_n$. If, instead of having $2n - 1$ (*i.e.*, $2n + 1 + 1 - 3$) consecutive colors, we have the same number of colors possibly not consecutive (because the colors around $l(c)$ are forbidden), *a fortiori* they will be sufficient for $L(3,2,1)$-labeling $v_1, \ldots v_n$.

So, consider palette $P = \{0, 1, \ldots, 2n+3\}$ and remove from it colors $l(c) - 2$, $l(c) - 1$, $l(c)$, $l(c) + 1$, and $l(c) + 2$ whenever they are in P because these are all the colors that cannot be used to label $v_1, \ldots v_n$. At least $n + 4 - 5$ colors remain available for $v_1, \ldots, v_n$, that are all at mutual distance 2 *via c*. Hence, the procedure that starts from the first available color and then hops 4 by 4 on the colors remaining in P after removing $f(c)$ and the colors too close to it produces a feasible $L(3,2,1)$-labeling.

The equality holds given reasonings analogous to those used to justify Lemma 1, together with the hypothesis that $l(c)$ eliminates from the palette exactly 5 colors.

If $n \leq 4$, the reasoning is the same, but the labeling is different. We omit the details here not to overburden the exposition, but they can be easily deduced from the proof of the previous lemma. □

We conclude this section by giving a general lower bound that, besides being of interest in itself, will be exploited when dealing with the considered subclasses of planar graphs.

Theorem 3. *For any graph G with maximum degree $\Delta \geq 2$, if a maximum degree node with two neighbors of degree Δ exists, then $\lambda(G) \geq 2\Delta + 2$.*

Proof. Let v be the node of degree Δ, and u and w the nodes adjacent to v and of degree Δ, too.

Consider a labeling function for G, and particularly the label $l(v)$ assigned to v. If $l(v)$ is different from 0, 1, $\lambda(G)-1$ and $\lambda(G)$, then the label of v excludes 5 labels from the palette used to label all its neighbors; each one of the Δ v's neighbors have a different color, and these must be at a mutual distance of two; since $l(v)$ could lay between two colors assigned to two of the v's neighbors, the number of used colors cannot be less than $\Delta+\Delta-2+5$, obtaining $\lambda(G) \geq 2\Delta+2$.

If, on the contrary, the label assigned to v belongs to $\{0, 1, \lambda(G)-1, \lambda(G)\}$, then consider node u. If the label assigned to u is different from 0, 1, $\lambda(G)-1$ and $\lambda(G)$, repeat the previous reasoning, getting $\lambda(G) \geq 2\Delta+2$; otherwise, since both v and u are nodes at a distance two from w with labels in $\{0, 1, \lambda(G)-1, \lambda(G)\}$, then necessarily w has a label outside this set. Hence, repeat the previous reasoning on w. □

3 Hexagonal Grids

It is known that $\lambda(G_4) = 2\Delta + 3 = 11$ [7] and $\lambda(G_8) = 23$ [4]; more recently, it has been proved that $\lambda(G_6) = 19$ [9]. So, in the following, we study the remaining grid G_3, closing the problem of $L(3, 2, 1)$-labeling for all the infinite regular grids.

Theorem 4. $\lambda(G_3) = 2\Delta + 3 = 9$.

Proof. We first prove that $\lambda(G_3) \geq 9$.

Preliminarily, observe that G_3 satisfies the hypothesis of Theorem 3 implying only $\lambda(G_3) \geq 2\Delta + 2 = 8$, that is not enough, so we proceed differently.

Assume by contradiction that $\lambda(G_3) = 8$. Consider an optimal labeling l of span $\sigma(l, G_3) = \lambda(G_3)$. Pick a v labeled $l(v) = 0$. Then, in order to remain inside the palette $\{0, \ldots, 8\}$, there must be one neighbor v' of v labeled $l(v') \in \{3, 4\}$. We consider the two cases separately.

Case 1: if $l(v') = 4$, it is not possible to assign 3 labels at a mutual distance of at least two to the three neighbors of v'.

Case 2: if $l(v') = 3$, the three neighbors of v' must have labels 0, 6, and 8. So, there is a neighbor v'' of v' such that $l(v'') = 6$, but it is not possible to assign 3 labels at a mutual distance of at least two to the three neighbors of v''. It follows that in any case $\lambda(G_3) \geq 9$.

Now, we provide a feasible $L(3, 2, 1)$-labeling using $P = \{0, 1, \ldots, 8, 9\}$ and this will prove that $\lambda(G_3) \leq 2\Delta + 3$.

We exploit the general technique introduced in [6], consisting in labeling one hexagon and then shifting this labeling, adding a coefficient mod $2\Delta + 4$.

Refer to Fig. 1(a). To determine the labeling l of a first hexagon and the coefficients bb, br, and bl, shifting the labels toward the bottom, bottom-right and bottom-left, respectively, we call the nodes of a hexagon a, b, c, d, e, and f in this order; it is not restrictive to assume that $l(a) = 0$; moreover, it must hold:

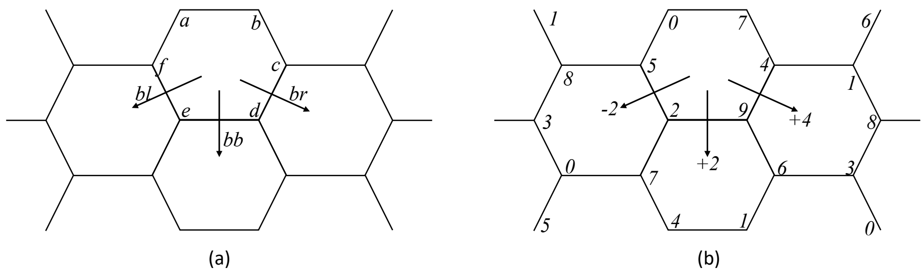

Fig. 1: (a) A portion of the hexagonal grid with the names associated with some of its nodes; (b) $L(3,2,1)$-labeling of a portion of the hexagonal grid.

- $l(e) = l(a) + bb = bb$ and $l(d) = l(b) + bb$;
- $l(f) = l(b) + bl$ and $l(e) = l(c) + bl$;
- $l(c) = l(a) + br = br$ and $l(d) = l(f) + br$;

where the sums are mod 10.

Since we want to shift the same labeling all over the grid, we implicitly consider cyclic distance; in other words, 0 and $\lambda(G_3)$ are considered adjacent labels.

Adding all the constraints required by the $L(3,2,1)$-labeling, and thanks to exhaustive reasoning, there are two possible solutions. The first one initially labels the nodes a, b, c, d, e, and f of the first hexagon with labels 0, 7, 4, 9, 2, 5 in this order, and chooses +2, +4, and -2, as coefficients shifting the labels toward the bottom, bottom-right and bottom-left, respectively. The second one initially labels a, b, c, d, e, and f with 0, 3, 6, 1, 8, 5, respectively, and chooses $bb = -2$, $br = +6$ and $bl = +2$. Without loss of generality, we focus on the first solution, leading to the labeling shown in Fig. 1(b).

This method produces a labeling containing a pattern repeated along the grid; it is easy to verify that it is a feasible (cyclic) $L(3,2,1)$-labeling of G_6 with span 9. □

It is worth noting that the above proof of the lower bound provides $\lambda(G) \geq 9$ for every graph of minimum degree 3, including 3-regular graphs, that is a bound better than the ones given by Theorems 1 and 3.

Corollary 1. *$\lambda^c(G_3) = 9$.*

It is worth noting that even the result for squared and octagonal grids produces a cyclic labeling.

4 Square of cycles

It is known [8] the exact value of λ for cycles. Namely, when $n \geq 8$, $\lambda(C_n) = 7$ if n is even, while $\lambda(C_n) = 8$ if n is odd. It is worth mentioning that the labeling algorithms provided in [8] output an $L(3,2,1)$-labeling that is cyclic for the n long cycles with n odd, while it is not cyclic in the even case.

Here, we determine the $L(3,2,1)$-labeling number of the square of all cycles, although they are planar graphs only when n is even or $n = 3$.

Let us consider the small values of n first.

If $n = 3$, C_n and C_n^2 coincide with K_3, and hence $\lambda(C_n^2) = \lambda(K_3) = 3n - 3 = 6$. When $n = 4$ and $n = 5$, C_n^2 coincide with K_n and again $\lambda(C_n^2) = 3n - 3$.

In all these three cases, the $L(3,2,1)$-labeling of C_n^2 is not cyclic as, although only colors at a mutual distance three are used, labels 0 and λ are assigned to adjacent nodes.

When $n = 6$, each node v and its neighbors induce in C_n^2 a fan F_4, from which we easily deduce $\lambda(C_n^2) \geq \lambda(F_4) = 10$. Because of the generality of the choice of v, we can apply Theorem 2 so having $\lambda(C_6^2) \geq 11$. This lower bound is not tight, indeed, $\lambda(C_6^2) = 12$. We omit the proof due to space reasons.

Lemma 2. $\lambda(C_n^2) \geq 12$ *when* $n \geq 7$.

Proof. First, observe that if l assigns a color c to v_1, the same color cannot be assigned to another node v_i if $i < 8$ because v_1 is at a distance ≤ 3 from v_i for all $i < 8$. Hence we need at least 7 different colors to label the nodes of C_n^2.

Assume by contradiction that there exists a feasible $L(3,2,1)$-labeling l of C_n^2 with span $\sigma(l, C_n^2) = 11$.

Let $l(v_1) = 4$. Then, its four neighbors (that are at a mutual distance of at least two) are forced to receive colors 7, 9, 11, and one between 0 and 1 in some order. We list all the possibilities:

- if $l(v_2) = 9$, then $l(v_3)$ is neither 7 nor 11 and will be either 0 or 1; v_n is adjacent to both v_1 and v_2 so also $l(v_n)$ is neither 4 nor 7 nor 11 and can only be either 0 or 1; since v_3 and v_n are at distance 2, this configuration is unfeasible;
- if $l(v_3) = 9$, then $l(v_2)$ is neither 7 nor 11, so it must be either 0 or 1; in this way, colors 7 and 11 will be assigned to v_n and v_{n-1} in some order. In both cases, $l(v_{n-2})$ is necessarily equal to 2 (if $l(v_2) = 0$, indeed if $l(v_{n-2}) = 1$, no possibility remains for $l(v_{n-2})$); no available colors are for $l(v_{n-3})$.
- analogously, we discard the possibilities of assigning color 9 to either v_{n-1} or v_n because of the symmetry between v_2, v_3 and v_n, v_{n-1} w.r.t. v_1.

Given the possibility of shifting v_1 in any position of C_n^2, we conclude that label 4 cannot be assigned to any node.

Let now $l(v_1) = 9$. Then, its four neighbors are forced to receive colors 0, 2, 4, and 6 in some order. Nevertheless, label 4 has already been excluded, so it is impossible to conclude a feasible labeling. Hence, label 9 cannot be assigned to any node.

If $l(v_1) = 6$, then its four neighbors are forced to receive colors 9, 11, one between 0 and 1, and one between 2 and 3. But we already know that 9 cannot be assigned to any node, so also label 6 leads to an unfeasible situation and hence cannot be assigned to any node.

Given the symmetry of the palette, it follows that also colors $\sigma(l, C_n^2) - 9 = 2$, $\sigma(l, C_n^2) - 6 = 5$, and $\sigma(l, C_n^2) - 4 = 7$ cannot be assigned to any node; otherwise, we could start from a feasible $L(3,2,1)$-labeling l using labels 2, 5, and 7 and get a feasible $L(3,2,1)$-labeling l' using labels 4, 6, and 9.

In view of the previous reasonings, if it were $\lambda(C_n^2) \leq 11$, the remaining colors would be only six (*i.e.*, 0, 1, 3, 8, 10 and 11) and hence not enough, implying a contradiction, hence $\lambda(C_n^2) \geq 12$. □

The previous lower bound is tight in some cases, as shown in the following result. Nevertheless, there are some values of n for which more colors are necessary (see Lemma 4).

Lemma 3. $\lambda(C_n^2) = 12$ *when* $n \equiv 0 \mod 7$.

Proof. The lower bound comes from Lemma 2, while the upper bound derives from the following labeling:

$$l(v_{7i+1}) = 0, l(v_{7i+2}) = 4, l(v_{7i+3}) = 8,$$

$$l(v_{7i+4}) = 12, l(v_{7i+5}) = 2, l(v_{7i+6}) = 6, l(v_{7i+7}) = 10,$$

for each $i = 0, \ldots \frac{n}{7} - 1$. It is immediate to check the feasibility of this labeling. □

Note that the labeling provided in the previous proof is not cyclic.

Lemma 4. $\lambda(C_{11}^2) = 15$ *and* $\lambda(C_{12}^2) = 16$.

Proof. To prove the lower bound, observe that both C_{11}^2 and C_{12}^2 are diameter 3 graphs, so every node must receive a different color. W.l.o.g. let $l(v_1) = x$ for some x in the palette; only v_6, v_7 and v_8 are at distance 3 from v_1 in C_{12}^2, and only v_6 and v_7 are at distance 3 from v_1 in C_{11}^2; so, if color $x+1$ is assigned to some node, it is one of them.

In all cases, it is not difficult to see that color $x+2$ cannot be assigned to any other node (because it is too close either to v_1 or to the node labeled with $x+1$), and hence $x+2$ is unused.

For the generality of the choice of x, we can say that for any two consecutive used colors, the next one must remain unused; in other words, it is possible to use at most two colors out of any three consecutive ones, so $\lambda(C_{11}^2) \geq 15$ and $\lambda(C_{12}^2) \geq 16$.

The upper bounds follow by labelings the nodes of C_{12}^2 with the sequence $0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13, 16$ and the nodes of C_{11}^2 with the sequence $0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13$. □

Lemma 5. $\lambda(C_n^2) \leq 14$ *when* $n = 7m + q$ *with* $m \geq q \geq 1$.

Proof. The upper bound on λ derives from the labeling that repeats the sequence $0, 4, 8, 12, 2, 6, 10$ for exactly m times; for q times, between two consecutive 7-long sequences, color 14 is used. It is immediate to check the feasibility of this labeling. □

We conclude this section with the following summarizing theorem, covering all large values of n:

Theorem 5. $12 \leq \lambda(C_n^2) \leq 14$ *when* $n \geq 42$ *and* $\lambda(C_n^2) = 12$ *when* $n \equiv 0 \mod 7$.

It is worth to be noted that we have got exact results for many small values of n, but we omit them here due to space reasons.

5 Outerplanar graphs

In this section, we propose an $L(3, 2, 1)$-labeling algorithm for outerplanar graphs that provably uses a linear number of colors in Δ. Due to space reasons, we omit the proof, that we move to the Appendix.

Consider an embedding of an outerplanar graph $G = (V, E)$, choose a node r, and induce a total order on the nodes by walking clockwise around the external face. Compute a Breadth First Search starting from node r so that nodes coming first in the ordering are visited first. As in [6], in the following, such computation will be called *Ordered Breadth First Search (OBFS)* while *Ordered Breadth First Tree (OBFT)* is the (unique) tree resulting from this special kind of breadth first search (for an example, see Fig. 2.b). The left-to-right order on each layer l of the OBFT induces a numbering of the nodes: we will call $v_{l,i}$ a node of G lying on layer l of the tree and occupies the i-th position in the left-to-right ordering on the layer (see Fig. 2.c).

In [6], an attractive property has been introduced for the OBFT of an outerplanar graph, extending the very well-known one holding for every BFT.

Lemma 6. *Let* $(v_{l,h}, v_{l',k})$*,* $l' \leq l$*, be a non-tree edge in an OBFT of an outerplanar graph* G*. Then:*

- *either* $l' = l$ *and (if, w.l.o.g.,* $k > h$*)* $k = h + 1$ *– see,* e.g.*, edges* $(v_{4,1}, v_{4,2})$ *and* $(v_{3,3}, v_{3,4})$ *in Fig. 2.c;*
- *or* $l' = l - 1$ *and* $k = r + 1$*, where* r *is the index of the parent of* $v_{l,h}$ *at layer* $l - 1$*; moreover,* $v_{l,h}$ *is the rightmost child of* $v_{l-1,r}$ *(see,* e.g.*, edges* $(v_{5,2}, v_{4,2})$ *and* $(v_{3,3}, v_{2,3})$ *in Fig. 2.c).*

Given an outerplanar graph $G = (V, E)$ and a palette P, a greedy coloring algorithm able to produce a feasible $L(3, 2, 1)$-labeling for G is the following:

`Algorithm GreedyLabelOuterplanarGraphs`
Input: an outerplanar graph $G = (V, E)$
Output: an $L(3, 2, 1)$-labeling f for G
`choose a node as` r`;`

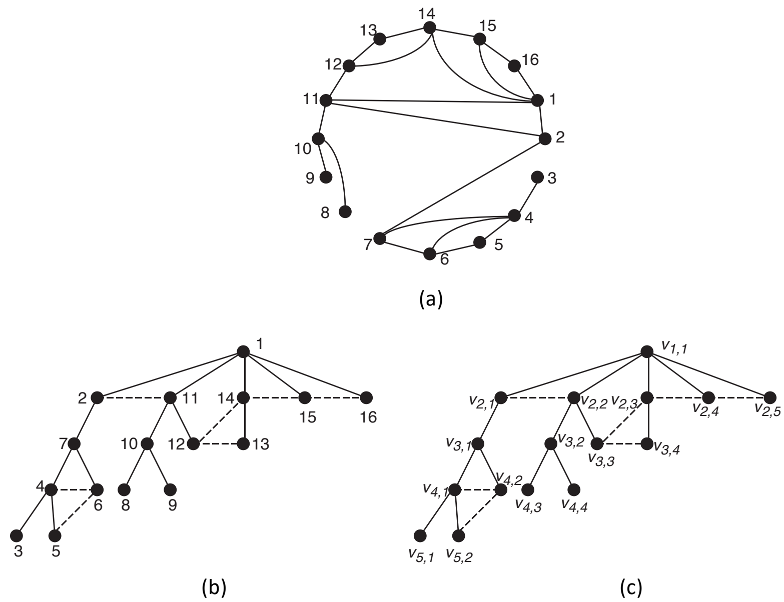

Fig. 2: An outerplanar graph and its OBFT.

```
compute an OBFS of G and generate its OBFT T;
label the nodes of the subgraph induced by r and its children, that
is, a subgraph of a fan, according to Lemma 1;
for each layer l from the root to the leaves of uncolored nodes:

   for each node v_{l,k} from left to right:
    • let S_{l,k} be the set of v_{l,k}'s children;
    • consider the subgraph induced by v_{l,k} and S_{l,k};
    • remove from P all the colors that cannot be used for labeling
      any node of S_{l,k} because of a too close already labeled node;
    • label set S_{l,k} according to the proof of Theorem 2 with the first
      colors remained in P;
    • restore P with all colors;

return the L(3,2,1)-labeling f for G.
```

Theorem 6. *Algorithm* GreedyLabelOuterplanarGraphs *correctly* $L(3,2,1)$-*labels an outerplanar graph with a span of at most* $4\Delta + 14 + K$ *if* $\Delta \geq 9$, *and of at most* $4\Delta + 15 + K$ *if* $\Delta \leq 8$, *where* K *is a constant upper bounded by 12.*

In this conference version, we omit the (long) proof of this theorem due to space limits, It is worth noticing that it was not our aim to determine a tight value for λ (a not complex though intricated issue) but to prove that it is linear in Δ.

6 Conclusions

In this paper, the $L(3,2,1)$-labeling problem on some subclasses of planar graphs is tackled, adding some relevant pieces to the general picture concerning this problem. Indeed, while the general upper bound on λ is quadratic in the maximum degree, it comes out to be linear in Δ for all the considered graph classes.

In particular, the exact $L(3,2,1)$-number of infinite hexagonal grids is determined in Section 3. Then, the problem on the square of n cycles is approached in Section 4, determining close upper and lower bounds on λ when n is large enough; for some special values of n the exact value of λ is determined. Finally, the $L(3,2,1)$-labeling on outerplanar graphs is studied in Section 5, providing even in this case a linear upper bound on λ.

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