# Utility-Sharing Games: How to Improve the Efficiency with Limited Subsidies*

Vittorio Bilò[1], Lucaleonardo Bove[2,3], and Cosimo Vinci[1]

[1] Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy
[2] Dipartimento di Ingegneria dell'Innovazione, Università del Salento, Lecce, Italy
[3] CAMPI Lab - Centre of Mathematics and Physics for Industry, Lecce, Italy
{vittorio.bilo,lucaleonardo.bove,cosimo.vinci}@unisalento.it

**Abstract.** In this work, we consider the problem of improving the efficiency of utility-sharing games, by resorting to a limited amount of subsidies. Utility-sharing games model scenarios in which strategic and self-interested players interact with each other by selecting resources. Each resource produces a utility that depends on the number of players selecting it, and each of these players receives an equal share of this utility. As the players' selfish behavior may lead to pure Nash equilibria whose total utility is sub-optimal, previous work has resorted to subsidies, incentivizing the use of some resources, to contrast this phenomenon.
In this work, we focus on the case in which the budget used to provide subsidies is bounded. We consider a class of mechanisms, called $\alpha$-subsidy mechanisms, that allocate the budget in such a way that each player's payoff is re-scaled up to a factor $\alpha \geq 1$. We design a specific sub-class of $\alpha$-subsidy mechanisms, that can be implemented efficiently and distributedly by each resource, and evaluate their efficiency by providing upper bounds on their price of anarchy. These bounds are parametrized by both $\alpha$ and the underlying utility functions and are shown to be best-possible for $\alpha$-subsidy mechanisms. Finally, we apply our results to the particular case of monomial utility functions of degree $p \in (0, 1)$, and derive bounds on the price of anarchy that are parametrized by $p$ and $\alpha$.

**Keywords:** Utility Games · Resource Allocation · Subsidy Mechanisms · Pure Nash Equilibrium · Price of Anarchy

## 1 Introduction

In several real-life contexts arising from economics, operation research and computer science, we face the necessity of allocating a set of utility-producing resources to agents, in such a way that the total utility is maximized. For example, we could consider a scenario, connected with management engineering, in which each resource models a project or a task to be completed, and each agent is

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an employee of a company, or a server in a content delivery network, that can be assigned to one of the tasks. It is reasonable to assume that, the more the number of employees assigned to a task, the more the quality of the completed task (or the lower the completion time, or the higher the probability that the task will be correctly completed). Indeed, the employees assigned to the same task can work in team, and it is expected that the resulting quality improves as the working team includes new members.

When completed, each task generates a profit (i.e., a utility) that is proportional to the resulting quality, and this profit (or a percentage of it) is equally shared among the employees who contributed to the task. As the number of employees and tasks could be very high, the presence of a centralized coordinator imposing all the assignments might be impracticable. Therefore, a decentralized implementation of the system, where each worker autonomously decides which task she wants to contribute to, is a more reasonable choice. To describe the effects of decentralization, we consider a game representation of the system in which each worker acts as a player who aims at maximizing the fraction of the profit that she receives (i.e., her payoff). This creates an interplay of strategic behavior, in which players compete with each other by selecting the tasks (i.e., the resources) that maximize their payoff. This may lead to suboptimal outcomes, in which the total utility is lower than the one achievable by a central authority imposing an optimal assignment of players to resources.

*Algorithmic game theory* [41] offers several tools to describe how the strategic choices of the players may affect the total utility achieved by all resources. First, the notion of *pure Nash equilibrium* [40], that is an outcome in which no player can increase her payoff by unilaterraly deviating to another strategic choice, is used to model stable solutions arising from selfish behavior. Then, the *Price of Anarchy* [36], which compares the total utility of any pure Nash equilibrium against the optimal total utility achievable in a centralized and coordinated environment, is adopted to quantify the lack of cooperation and coordination.

**Our Contribution.** Given the difficulties in coordinating the players' strategic behavior, a reasonable approach to convey them toward better pure Nash equilibria is that of providing subsidies encouraging the use of certain resources. Several works [3, 21, 22, 29, 35, 44–46] showed the effectiveness of this idea, by designing ad-hoc subsidy allocation mechanisms that are able to improve the price of anarchy. The amount of subsidies that these mechanisms require, however, can be very high, thus limiting their applicability to most real-life contexts, where budgets are usually severely constrained.

In this work, we show how to improve the efficiency of decentralized allocation systems, when the total amount of subsidies available to each resource is somewhat constrained by the total utility that can be generated by the resource itself. We model allocation systems as a class of games, called *utility-sharing games*, which constitutes a subclass of the general framework of *monotone valid utility games* defined in [53], and is similar and/or equivalent to other game classes studied in [4, 15, 32, 33, 35, 38, 39]. In utility-sharing games, we have a finite set of players, a finite set of resources available to the players, and each resource is associated with a certain *utility function* whose value is equally shared among the players selecting it. Each player aims at maximizing her *payoff*, given by the fraction of utility she receives.

The main novelty of this work is the design and the analysis of *$\alpha$-subsidy mechanisms* ($\alpha$-SMs), a new class of subsidy allocation mechanisms that, parametrized by a value $\alpha \geq 1$, allocate to each resource an amount of subsidies that is at most $\alpha - 1$ times the utility produced by the resource, so that the players' payoffs can be re-scaled up by a multiplicative factor $\alpha$.

We provide tight bounds on the price of anarchy guaranteed by $\alpha$-SMs for several classes of utility-sharing games. In particular, we resort to a particular sub-class of $\alpha$-SMs, called *optimal-congestion-based* $\alpha$-SMs, that can be computed and executed in polynomial time (Theorem 1), and we provide upper bounds on the resulting price of anarchy that depend on $\alpha$, the number of players $n$, and the class of utility functions of the game (Theorem 2); we also provide simpler bounds that depend on $n$ and $\alpha$ only (Corollary 1).

Conversely, we show that optimal-congestion-based mechanisms achieve best-possible performances within the general class of $\alpha$-SMs, that is, no $\alpha$-SM can further lower our bounds on the price of anarchy (Theorem 3, Corollaries 2 and 3). Finally, we apply our general results to the specific case of utility functions representable as monomials of fixed degree $p \in [0, 1]$ (Theorem 4).

We point out that, for any utility-sharing game and sufficiently large $\alpha \geq 1$, all pure Nash equilibria induced by optimal-congestion-based $\alpha$-SMs maximize the total utility (Remark 1). Thus, our approach guarantees the same performance of the subsidy allocation mechanisms studied in [35], that, differently from ours, may fail under some budget limitations. Furthermore, for $\alpha = 1$, we re-obtain the tight bounds on the price of anarchy for utility-sharing games without subsidies (Remark 2), already shown in [35, 38].

**Further Related Work.** The first general game-theoretic model for decentralized resource allocation systems with payoff-maximizing players is that of monotone valid utility games [53], where the payoff functions satisfy some mild assumptions, such as monotonicity and submodularity w.r.t. the selected resources. In this seminal paper, a tight bound of 2 on the price of anarchy of monotone valid utility games is provided. Subsequently, several (sub)classes of monotone valid utility games have been introduced and studied.

A work that is strictly close to ours is [38], which studies the price of anarchy of (an equivalent model of) utility-sharing games, and provided tight bounds that are parametrized by the number of players and the considered utility functions; tight bounds on the price of anarchy for more general settings in which the set of available resources is player-specific is also provided. Papers [15, 35] model strategic project selection as specific instantiations of monotone valid utility games, and provide more specific bounds on the price of anarchy and other efficiency metrics (such as the price of stability [2]). In [33, 39], the efficiency of specific monotone valid utility games where, differently from our model of utility-sharing games the sharing rules do not necessarily split each resource utility in an equal way among the players selecting it, has been considered.

The problem of determining mechanisms improving the price of anarchy in utility-sharing games (so as for their variants and/or generalizations) has been widely considered in the literature. In [35], it is shown how to assigns a credit (i.e., a subsidy) to each project, so as to guarantee that any pure Nash equilibrium is an optimal strategy profile. A considerable amount of work [21, 22, 29, 44–46] shows how to modify the payoffs of the players participating in utility-sharing games (e.g., via subsidies), with the purpose of improving the efficiency of pure Nash equilibria; in particular, tight bounds on the resulting price of anarchy, that depend on the considered class of utility functions, are provided.

Utility-sharing games are strictly related to the cost-minimization game-theoretic model of *congestion games* [16, 48]. Congestion games are resource selection games with a finite set of cost-minimizing players and a finite set of resources, where each player selects a subset of resources (among a finite collection that is player-specific), and the cost of each selected resource is a function of the number of players selecting it. In particular, utility-sharing games can be seen as the payoff-maximization version of congestion games with symmetric players [30, 37] (that is, all players can share all resources) and singleton strategies (that is, each player can select exactly one resource). The problem of measuring the price of anarchy of congestion games has been a hot-topic in algorithm game theory in the last two decades [5, 8, 23, 51], and several works have provided upper and lower bounds depending on the considered cost-functions [1, 7, 9, 11, 17, 30, 34, 50] or the structure of the players' strategies [6, 7, 10, 11, 14, 18, 27, 30, 37]. Furthermore, several works have also focused on the design and analysis of mechanisms to improve the price of anarchy. The following classes of mechanisms have been widely studied: *taxation mechanisms* [12, 20, 26, 29, 42, 43, 47], where each player, instead of receiving a subsidy, is charged a tax; *Stackelberg strategies* [13, 30, 49, 52], in which a fraction of the players can be controlled by a central authority; *coordination-mechanisms* [19, 24, 25], where the order in which players are processed is decided by a local policy implemented on each resource; *cost-sharing mechanisms* [19, 28, 31], that decide the rules to share the cost of each resource among the players selecting it.

## 2 Model and Definitions

Given an integer $k \geq 1$, let $[k] := \{1, 2, \ldots, k\}$ denote the set of the first $k$ positive integers. Given an integer $h \geq 0$, $\mathbb{N}_{\geq h}$ denotes the set of natural numbers higher or equal to $h$, and $\mathbb{N} := \mathbb{N}_{\geq 1}$ denotes the set of natural numbers.

**Utility-Sharing Games.** A *utility-sharing game* is formally defined as a tuple $\mathrm{SG} = (N, R, (u_r)_{r\in R})$ where $N = [n]$ is a set of $n$ players, $R = \{r_1, \ldots, r_m\}$ is a set of $m$ *resources* and $u_r : \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a non-negative *(resource) utility function* associated with each resource $r \in R$. We further assume that (i) $u_r(0) = 0$, (ii) $u_r(x) > 0$ for some $x \in \mathbb{N}$, and $u_r$ is (iii) non-decreasing and (iv) concave in $\mathbb{N}_{\geq 0}$; in particular, (i) holds since a resource without players does not produce any utility, (ii) has been considered to avoid the presence of resources that do not produce any utility, (iii) holds since the more the players, the higher the utility, and (iv) holds since the contribution of a player who joins a resource decreases as the congestion of that resource increases.

A *strategy profile* (or *assignment*) $\boldsymbol{\sigma} = (\sigma_i)_{i\in N}$ is a configuration in which each player $i$ has selected resource $r = \sigma_i \in R$, and $\sigma_i$ denotes the *strategy* of player $i$ in $\boldsymbol{\sigma}$. The *congestion* $n_r(\boldsymbol{\sigma}) := |\{i \in N : \sigma_i = r\}|$ of resource $r$ in strategy profile $\boldsymbol{\sigma}$ is the total amount of players selecting $r$ in $\boldsymbol{\sigma}$. Given a strategy profile $\boldsymbol{\sigma}$, the *payoff* of player $i$ is defined as $p_i(\boldsymbol{\sigma}) := u_{\sigma_i}(n_{\sigma_i}(\boldsymbol{\sigma}))/n_{\sigma_i}(\boldsymbol{\sigma})$. Informally, we assume that the utility achieved on each resource is equally shared among the players selecting it, and this fraction of utility determines the payoff of each player. The *total utility* function $\mathrm{SUM}(\boldsymbol{\sigma}) := \sum_{r\in R} u_r(n_r(\boldsymbol{\sigma}))$ is equal to the sum of all resource utilities. We have that $\mathrm{SUM}(\boldsymbol{\sigma}) = \sum_{r\in R} u_r(n_r(\boldsymbol{\sigma})) = \sum_{r\in R} |\{i \in N : \sigma_i = r\}| \cdot \frac{u_{\sigma_i}(n_{\sigma_i}(\boldsymbol{\sigma}))}{n_{\sigma_i}(\boldsymbol{\sigma})} = \sum_{i\in N} p_i(\boldsymbol{\sigma})$, that is, the total utility can be seen as the sum of all payoffs, i.e., it is a social welfare function that is proportional to the overall satisfaction of all players.

**Pure Nash Equilibria and Price of Anarchy.** All players aim at maximizing their payoffs, regardless of the others. As a reasonable outcome of selfish behavior, we consider the notion of *pure Nash equilibrium* [40]. Given a strategy profile $\boldsymbol{\sigma}$, a player $i \in N$ and a resource $r$, let $(\boldsymbol{\sigma}_{-i}, r)$ denote the strategy profile $\boldsymbol{\sigma}'$ in which player $i$ chooses resource $r$ (that is, $\sigma'_i = r$), and all other players choose the same resource as in $\boldsymbol{\sigma}$ (that is, $\sigma'_h = \sigma_h$ for any $h \in N \setminus \{i\}$). A strategy profile $\boldsymbol{\sigma}$ is a *pure Nash equilibrium* if, for any player $i$ and resource $r \in R$, we have $p_i(\boldsymbol{\sigma}) \geq p_i(\boldsymbol{\sigma}_{-i}, r)$, that is, no player improves her payoff by deviating to another resource. Given a utility-sharing game SG, let SP(SG) denote the set of strategy profiles of SG, and PNE(SG) denote the set of pure Nash equilibria of SG; furthermore, let $\mathrm{OPT(SG)} = \max_{\boldsymbol{\sigma}\in SP(\mathrm{SG})} \mathrm{SUM}(\boldsymbol{\sigma})$ denote the maximum total utility achievable in SG.

A universal metric to measure the impact of selfishness on the total utility is the *price of anarchy* [36]. Given a utility-sharing game SG, the *price of anarchy* (PoA) of SG is defined as $\mathrm{PoA(SG)} = \max_{\boldsymbol{\sigma}\in \mathrm{PNE(SG)}} \frac{\mathrm{OPT(SG)}}{\mathrm{SUM}(\boldsymbol{\sigma})}$, that is, the worst-case ratio between the optimal total utility and that achieved by any pure Nash equilibrium of SG. We observe that, the lower the price of anarchy, the lower the impact of selfish behavior in terms of total utility.

**Subsidy Mechanisms.** We assume that each resource can implement a local policy that, by means of a subsidy, may increase the payoffs of players selecting it, up to a maximum factor $\alpha \geq 1$. We refer to this set of local policies as *$\alpha$-subsidy mechanisms* ($\alpha$-SMs). In particular, an $\alpha$-subsidy mechanism $\Pi_\alpha$ takes as input a utility-sharing game SG and returns a new utility-sharing game $\mathrm{SG}_\alpha = (N, M, (u'_r)_{r\in M})$ that will be played in place of SG, where each utility function $u'_r$ is called *perceived utility* of resource $r$, and verifies the following properties: (i) $u'_r(x) = u_r(x) + \theta_r(x)$ for any $x \in \mathbb{N}_{\geq 0}$, for an opportune non-negative *subsidy function* $\theta_r$; (ii) the subsidy function $\theta_r$ is locally computed by resource $r$, based only on the knowledge of the initial game SG and the congestion $n_r(\boldsymbol{\sigma})$ of $r$ in a given strategy profile $\boldsymbol{\sigma}$; (iii) $\theta_r(x) \leq (\alpha-1)\cdot u_r(x)$, that is, $u'_r(x) \leq \alpha u_r(x)$, for any $x \in \mathbb{N}_{\geq 0}$. We observe that the perceived utilities $u'_i$ lead to a new players' payoff $p'_i(\boldsymbol{\sigma}) = (u_{\sigma_i}(n_{\sigma_i}(\boldsymbol{\sigma})) + \theta_r(n_{\sigma_i}(\boldsymbol{\sigma})))/n_{\sigma_i}(\boldsymbol{\sigma})$, for any strategy profile $\boldsymbol{\sigma}$ of $\mathrm{SG}_{\Pi_\alpha}$ and player $i \in N$.

In the following, we provide some justifications on the above properties characterizing $\alpha$-SMs. Property (i) states that the perceived utility is equal to the overall value given by the utility and the additional subsidy (determined by $\theta_r$) assigned by the mechanism to each resource; then, both the utility and the subsidy are shared by the players selecting the resource, thus leading to new players' payoffs. Property (ii) is motivated by the fact that the mechanism can be reasonably executed in a distributed way, where each resource uses its local information on the congestion, without knowing the congestion of the other resources. Finally, property (iii) is motivated by scenarios in which the subsidies, because of some budget constraints, are limited by a factor $\alpha$ of the utility achievable by each resource.

Given an $\alpha$-SM $\Pi_\alpha$ for SG, the *price of anarchy of* SG *under* $\Pi_\alpha$ is defined as $\mathrm{PoA}(\mathrm{SG}, \Pi_\alpha) = \max_{\boldsymbol{\sigma}\in \mathrm{PNE}(\mathrm{SG}_\alpha)} \frac{\mathrm{OPT(SG)}}{\mathrm{SUM}(\boldsymbol{\sigma})}$, where $\mathrm{PNE}(\mathrm{SG}_\alpha)$ is the set of pure Nash equilibria of the game $\mathrm{SG}_\alpha$ induced by $\Pi_\alpha$, OPT(SG) is the optimal total utility of the initial game SG and $\mathrm{SUM}(\boldsymbol{\sigma})$ is the total utility computed according to the utility-functions of the initial game SG. Informally, the price of anarchy of a game SG under an $\alpha$-SM $\Pi_\alpha$ measures how bad is a pure Nash equilibrium of the game modified by $\Pi_\alpha$ compared with the optimal total utility, but considering as total utility functions those related to the initial game SG. The modelling choice for which subsidies are not taken into account in the total utility appearing in the price of anarchy is motivated by the realistic scenario in which players receive money as subsidies, and then the same amount of money is lost by the central authority (e.g., a governmental entity) who disburses them. Thus, the overall contribution to the social welfare of subsidies is null, that is, the social welfare continues to be equal to the total utility without subsidies.

## 3 Computation and Efficiency of $\alpha$-SMs

In this section, we first define a specific $\alpha$-SM that is polynomial-time computable and executable, and then we measure its efficiency by providing bounds on the resulting price of anarchy. We first provide some preliminary notation. We say that a function $f : \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *payoff-regular* if the function $u$ defined as $u(x) := x \cdot f(x)$ is a proper utility function (that is, $f$ determines the payoff associated with utility function $u$). Given a class of payoff-regular functions $\mathcal{D}$, let $\mathcal{L}(\mathcal{D}) := \{u : u(x) = w \cdot x \cdot f(x),\ f \in \mathcal{D},\ w > 0\}$ denote the family of utility functions whose related payoff functions, up to a scaling factor, belong to $\mathcal{D}$. The following proposition, whose proof is deferred to the full version of the paper, shows some useful properties of payoff-regular functions.

**Proposition 1.** *Each payoff-regular function is non-increasing and positive in* $\mathbb{N}$.

### 3.1 Computation

Fix $\alpha \geq 1$. Given a game $\mathrm{SG} = (N, M, (u_r)_{r\in M})$ and an optimal strategy profile $\boldsymbol{\sigma}^*$ of SG, let $\Pi_\alpha(\boldsymbol{\sigma}^*)$ be an $\alpha$-SM for SG that returns a new utility-sharing game $\mathrm{SG}_\alpha = (N, M, (u'_r)_{r\in M})$ with perceived utility functions $u'_r$ defined as follows. For any $x \in \mathbb{N}_{\geq 0}$ and strategy profile $\boldsymbol{\sigma}$:

$$u'_r(x) = \begin{cases} u_r(x), & \text{if } n_r(\boldsymbol{\sigma}) \geq n_r(\boldsymbol{\sigma}^*), \\ \alpha \cdot u_r(x), & \text{if } n_r(\boldsymbol{\sigma}) < n_r(\boldsymbol{\sigma}^*). \end{cases} \tag{1}$$

Such an $\alpha$-SM is called *optimal-congestion-based*. We observe that optimal-congestion-based $\alpha$-SMs encourage the use of resources whose congestion in the optimal strategy profile is higher than that in the played strategy profile, and this is done by increasing the utility of such resources by a factor of $\alpha$.

In the following theorem, we show that optimal-based-congestion $\alpha$-SMs, under mild assumptions, can be computed and executed in polynomial time.

**Theorem 1.** *Given $\alpha \geq 1$ and a utility-sharing game* SG *with utility functions in $\mathcal{L}(\mathcal{D})$, with $\mathcal{D}$ containing payoff-regular functions only, we can compute and execute in polynomial time an optimal-based-congestion $\alpha$-SM for* SG*.*

*Proof (Sketch).* We observe that, once an optimal strategy profile $\boldsymbol{\sigma}^*$ of SG is computed, then the $\alpha$-SM $\Pi_\alpha(\boldsymbol{\sigma}^*)$ can be executed in polynomial time. Thus, to show the claim, it is sufficient to show that an optimal strategy profile $\boldsymbol{\sigma}^*$ for SG can be computed in polynomial time. To compute an optimal strategy profile $\boldsymbol{\sigma}^*$ of SG, we can apply a greedy algorithm that processes all players following the ordering induced by their indices, and each processed player $i \in N$ is assigned to the resource $r$ that minimizes the quantity $u_r(k_{i-1,r}+1) - u_r(k_{i-1,r})$, where ties are broken in favor of the resource with lower index, and $k_{i-1,r}$ denotes the congestion of resource $r$ under the strategy profile obtained after assigning the first $i-1$ players. This greedy algorithm obviously runs in polynomial time.

The proof that the strategy profile returned by the greedy algorithm is optimal for SG is based on the concavity of the utility functions $u_r$, and is deferred to the full version. □

### 3.2 Efficiency

In the following, we provide an upper bound on the price of anarchy under optimal-congestion-based $\alpha$-SMs, that is parametrized by the considered class of utility functions; furthermore, we show that, under mild assumptions on the considered utility functions, no $\alpha$-SM can improve the efficiency achieved by optimal-congestion-based $\alpha$-SMs, that is, the latter are essentially the best possible $\alpha$-SMs mechanisms.

**PoA Upper Bounds.** We first give some preliminary notation. Given a payoff-regular function $f$, let

$$\beta_{n,\alpha}(f) := \sup_{x,y\in\mathbb{N}_{\geq 0}: n\geq x\geq y\geq 0, x>0} \frac{y\left(f(y) - \frac{1}{\alpha}f(x)\right)}{xf(x)};$$

we observe that $\beta_{n,\alpha}(f) \geq 1 - 1/\alpha$ (indeed, it is sufficient to set $x = y$ in the argument of the supremum to get a value equal to $1 - 1/\alpha$); furthermore, the denominator appearing in the right-hand part of the definition is always non-zero, as each utility function $f(x)$ is positive in $x > 0$ (by Proposition 1). Given a class of payoff-regular functions $\mathcal{D}$, let $\beta_{n,\alpha}(\mathcal{D}) := \sup_{f\in\mathcal{D}} \beta_{n,\alpha}(f)$ and $\beta_{\alpha}(\mathcal{D}) := \sup_{n\in\mathbb{N}} \beta_{n,\alpha}(\mathcal{D})$.

**Theorem 2.** *Let $\mathcal{D}$ be a class of payoff-regular functions. Given a game* SG *with at most $n \geq 2$ players and utility functions in $\mathcal{L}(\mathcal{D})$, and given an optimal-congestion-based $\alpha$-SM $\Pi_\alpha(\boldsymbol{\sigma}^*)$ for* SG*, we have that*

$$\mathrm{PoA}(\mathrm{SG}, \Pi_\alpha(\boldsymbol{\sigma}^*)) \leq \frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D}) \leq \frac{1}{\alpha} + \beta_{\alpha}(\mathcal{D}).$$

*Proof (Sketch).* Let $\mathrm{SG} = (N, M, (u_r)_{r\in M})$ be a utility-sharing game with utility functions in $\mathcal{L}(\mathcal{D})$, and let $\boldsymbol{\sigma}^*$ be a social optimum of SG. Let $\boldsymbol{\sigma}$ be a pure Nash equilibrium of the game $\mathrm{SG}_\alpha$ obtained after applying the $\alpha$-SM $\Pi_\alpha(\boldsymbol{\sigma}^*)$. Let $o_j$ and $k_j$ be the congestions of each resource $r_j \in R$ in $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$, respectively.

For each $r_j \in R$ such that $k_j \geq o_j$, we have that

$$o_j f_j(o_j) - \frac{1}{\alpha} o_j f_j(k_j) \leq k_j f_j(k_j) \beta_{n,\alpha}(\mathcal{D}), \tag{2}$$

by definition of $\beta_{n,\alpha}(\mathcal{D})$. On the other hand, for any $r_j \in R$ such that $o_j > k_j$, we have that

$$o_j f_j(o_j) - (o_j - k_j) f_j(k_j+1) \leq o_j f_j(o_j) - (o_j - k_j) f_j(o_j) = k_j f_j(o_j) \leq k_j f_j(k_j), \tag{3}$$

where $k_j f_j(k_j) \geq k_j f_j(k_j + 1) \geq k_j f_j(o_j)$ trivially holds if $k_j = 0$, and holds even if $k_j > 0$, since $f$ is payoff-regular (thus, non-increasing by Proposition 1) and $k_j + 1 \leq o_j$ (by the integrality of $k_j$ and $o_j$).

Now, we recall that each utility function $u_j := u_{r_j}$ of SG can be written as $u_j(x) := w_j \cdot x \cdot f_j(x)$, with $w_j > 0$ and $f_j \in \mathcal{D}$. Thus, as $\boldsymbol{\sigma}$ is a pure Nash equilibrium in the new game $\mathrm{SG}_\alpha$, we observe that the value of each utility function $u'_j := u'_{r_j}$ achieved when playing $\boldsymbol{\sigma}$ in $\mathrm{SG}_\alpha$ is equal to $u'_j(k_j) = w'_j \cdot x \cdot f_j(k_j)$, where

$$w'_j = \begin{cases} w_j, & \text{if } k_j \geq o_j, \\ \alpha \cdot w_j, & \text{if } k_j < o_j \end{cases}; \tag{4}$$

analogously, the payoff function of each player in the new game $\mathrm{SG}_\alpha$ when playing the equilibrium $\boldsymbol{\sigma}$ is equal to $w'_j \cdot f_j(k_j)$. Thus, since $\boldsymbol{\sigma}$ is an equilibrium in game $\mathrm{SG}_\alpha$ and all payoff-regular functions are non-increasing (by Proposition 1), for each pair of resources $(r_j, r_h)$, we have that

$$w'_j f_j(k_j) \geq w'_h f_h(k_h + 1). \tag{5}$$

Furthermore, as $\sum_{r_j\in R} k_j = |N| = \sum_{r_j\in R} o_j$, we have the following equality (whose proof is deferred to the full version):

$$\sum_{r_j\in R: k_j \geq o_j} (k_j - o_j) = \sum_{r_j \in R: o_j > k_j} (o_j - k_j). \tag{6}$$

By combining (5) and (6), we obtain the following inequality:

$$\sum_{j:k_j \geq o_j} (k_j - o_j)\, w'_j f_j(k_j) - \sum_{j:k_j < o_j} (o_j - k_j)\, w'_j f_j(k_j + 1) \geq 0. \tag{7}$$

By combining (2), (3) and (7), and by exploiting the definition of $w'_j$ given in (4), we get the following inequalities (a more detailed list of inequalities is deferred to the full version):

$$\sum_{j\in R} o_j w_j f_j(o_j)$$

$$\leq \sum_{j\in R} o_j w_j f_j(o_j) + \frac{1}{\alpha}\left[\sum_{j:k_j \geq o_j} (k_j - o_j)\, \overbrace{w'_j}^{=w_j} f_j(k_j) - \sum_{j:k_j < o_j} (o_j - k_j)\, \overbrace{w'_j}^{=\alpha w_j} f_j(k_j + 1)\right] \tag{8}$$

$$= \left[\sum_{j:k_j \geq o_j} w_j \left(o_j f_j(o_j) - \frac{1}{\alpha} o_j f_j(k_j) + \frac{1}{\alpha} k_j f_j(k_j)\right)\right]$$

$$+ \left[\sum_{j:k_j < o_j} w_j \left(o_j f_j(o_j) - (o_j - k_j)\, f_j(k_j + 1)\right)\right]$$

$$\leq \left[\sum_{j:k_j \geq o_j} w_j \left(\beta_{n,\alpha}(\mathcal{D}) k_j f_j(k_j) + \frac{1}{\alpha} k_j f_j(k_j)\right)\right] + \left[\sum_{j:k_j < o_j} w_j k_j f_j(k_j)\right] \tag{9}$$

$$\leq \left[\sum_{j:k_j \geq o_j} w_j \left(\frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D})\right) k_j f_j(k_j)\right] + \left[\sum_{j:k_j < o_j} w_j \left(\frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D})\right) k_j f_j(k_j)\right] \tag{10}$$

$$= \left(\frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D})\right) \sum_{j\in R} k_j w_j f_j(k_j), \tag{11}$$

where (8) follows from (7), (9) follows from (2) and (3), and (10) holds since $1/\alpha + \beta_{n,\alpha}(\mathcal{D}) \geq 1$ (indeed, we already observed at the beginning of this section that $1 - 1/\alpha \leq \beta_{n,\alpha}(\mathcal{D})$).

Therefore, from (11) we have that $\sum_{j\in R} o_j w_j f_j(o_j) \leq \left(\frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D})\right) \sum_{j\in R} k_j w_j f_j(k_j)$, and we can conclude that

$$\text{PoA}(\text{SG}, \Pi_\alpha(\boldsymbol{\sigma}^*)) = \frac{\sum_{j\in R} u_j(o_j)}{\sum_{j\in R} u_j(k_j)} = \frac{\sum_{j\in R} o_j w_j f_j(o_j)}{\sum_{j\in R} k_j w_j f_j(k_j)} \leq \frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D}).$$

□

*Remark 1 (Optimal PoA via α-SMs).* Given a class of payoff-regular functions $\mathcal{D}$ and a game $\text{SG} = (N, M, (u_r)_{r\in M})$ with utility functions in $\mathcal{L}(\mathcal{D})$, if we apply to SG an optimal-based-congestion $\text{SM}_\alpha$ with $\alpha > \frac{\max_{r\in R, l\in[n]}(u_r(l)/l)}{\min_{r\in R, l\in[n]}(u_r(l)/l)}$, we have that the resulting pure Nash equilibrium $\boldsymbol{\sigma}$ necessarily coincides with the optimal strategy profile $\boldsymbol{\sigma}^*$ which the $\mathrm{SM}_\alpha$ is based on (the proof of this property is deferred to the full version). Thus, the price of anarchy becomes 1, and we re-obtain the optimal performance of the subsidy mechanisms considered in [35].

The following corollary of Theorem 2 provides a tight bound on the price of anarchy that depends on the number of players only, under mild assumptions on the considered utility functions.

**Corollary 1.** *Let* SG *be a utility-sharing game with at most* $n \geq 2$ *players and utility functions in a class* $\mathcal{L}(\mathcal{D})$, *with* $\mathcal{D}$ *containing payoff-regular functions only. For any optimal-congestion-based* $\alpha$*-SM* $\Pi_\alpha(\boldsymbol{\sigma}^*)$ *for* SG, *we have that*

$$\mathrm{PoA}(\mathrm{SG}, \Pi_\alpha(\boldsymbol{\sigma}^*)) \leq 1 + \frac{1}{\alpha}\left(1 - \frac{1}{n}\right) \leq 1 + \frac{1}{\alpha}.$$

*Proof.* To prove the claim it is sufficient to show that

$$\frac{y\left(f(y) - \frac{1}{\alpha}f(x)\right)}{xf(x)} \leq 1 - \frac{1}{\alpha n} \quad \forall x, y : n \geq x \geq y \geq 0, x > 0. \tag{12}$$

Indeed, by combining the previous inequality with Theorem 2, we get

$$\mathrm{PoA}(\mathrm{SG}) \leq \frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D}) \leq \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha n}\right) = 1 + \frac{1}{\alpha}\left(1 - \frac{1}{n}\right).$$

If $y = 0$, inequality (12) trivially holds. If $y \geq 1$, we have that

$$\frac{y(f(y) - \frac{1}{\alpha}f(x))}{xf(x)} = \frac{yf(y) - \frac{1}{\alpha}yf(x)}{xf(x)} \leq \frac{xf(x) - \frac{1}{\alpha}yf(x)}{xf(x)} = 1 - \frac{y}{\alpha x} \leq 1 - \frac{1}{\alpha n}, \tag{13}$$

where the first inequality follows from the fact that the functions of the type $t \cdot f(t)$ are necessarily non-decreasing (as they belong to a class $\mathcal{L}(\mathcal{D})$ that contains, by assumptions, proper utility functions only) and (13) holds since $y \geq 1$ and $x \leq n$. □

**PoA Lower Bounds.** In the following theorem, whose full proof is deferred to the full version, we show that no $\alpha$-SM can outperform the worst-case efficiency achieved by optimal-based-congestion $\alpha$-SMs.

**Theorem 3.** *Let* $\mathcal{D}$ *be a class of payoff-regular functions. For any* $\epsilon > 0$ *and* $n \in \mathbb{N}_{\geq 2}$, *there exists a utility-sharing game* SG *with at most* $n$ *players such that, for any* $\alpha$*-SM* $\Pi_\alpha$ *for* SG, *the price of anarchy of* SG *under* $\Pi_\alpha$ *is higher than* $1/\alpha + \beta_{n,\alpha}(\mathcal{D}) - \epsilon$.

*Proof (Sketch).* Fix $\epsilon > 0$ and $n \in \mathbb{N}_{\geq 2}$. By definition of supremum, we have that there exist $x, y \in \mathbb{N}_0$ with $n \geq x \geq y$ and $x > 0$, and $f \in \mathcal{D}$ such that $\frac{1}{\alpha} + \frac{y(f(y) - \frac{1}{\alpha}f(x))}{xf(x)} > 1 + \beta_{n,\alpha}(\mathcal{D}) - \epsilon$. Consider a game $\mathrm{SG} = (N, R, (u_r)_{r \in R})$ with $x = |N| \leq n$ players and a set $R$ of $x - y + 1$ resources. Set $R' := R \setminus \{r_1\} = \{r_2, \ldots, r_{x-y+1}\}$. The utility function of resource $r_1$ is $u_1(t) := w_1 \cdot t \cdot f(t)$, with $w_1 := 1$, while the utility function of each resource $r_j \in R'$ is $u_j(t) := u_2(t) =:$

$w_2 \cdot t \cdot f(t)$, with $w_2 := \frac{f(x)}{\alpha f(1)}$. One can show that, for any $\alpha$-SM $\Pi_\alpha$, the strategy profile $\boldsymbol{\sigma}$ in which all players choose resource $r_1$ is a pure Nash equilibrium under the application of $\Pi_\alpha$. Now, let $\boldsymbol{\sigma}^*$ be the strategy profile in which $y$ players choose resource $r_1$, while each of the remaining $x-y$ players chooses a different resource in $R'$. By estimating the ratio between the total utilities of $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$, the claim follows. □

We also have the following corollary of Theorem 3, that provides a lower bound that does not depend on the maximum number of players (the proof is deferred to the full version).

**Corollary 2.** *Let $\mathcal{D}$ be a class of payoff-regular functions. For any $\epsilon > 0$, there exists a utility-sharing game* SG *such that, for any $\alpha$-SM $\Pi_\alpha$ for* SG*, the price of anarchy of* SG *under $\Pi_\alpha$ is higher than $1/\alpha + \beta_\alpha(\mathcal{D}) - \epsilon$.*

Finally, the following corollary of Theorem 3 and Corollary 2 shows that the upper bound provided in Corollary 1 is tight (the proof is deferred to the full version).

**Corollary 3.** *(i) For any $n \in \mathbb{N}_{\geq 2}$, there exists a utility-sharing game* SG *with at most $n$ players such that, for any $\alpha$-SM $\Pi_\alpha$ for* SG*, the price of anarchy of* SG *under $\Pi_\alpha$ is at least $1 + \frac{1}{\alpha}\left(1 - \frac{1}{n}\right)$. (ii) Furthermore, for any $\epsilon > 0$, there exists a game* SG *such that the price of anarchy of* SG *under $\Pi_\alpha$ is higher than $1 + \frac{1}{\alpha} - \epsilon$.*

*Remark 2 (PoA without subsidies).* If $\alpha = 1$, an optimal-congestion-based $\alpha$-SM does not change the utility functions of the original game. Thus, the tight bounds provided in Theorems 2-3 and Corollaries 1-3 with $\alpha = 1$ are also tight bounds on the price of anarchy of utility-sharing games without the use of any subsidy mechanism, that is, we can re-obtain the tight bounds on the price of anarchy provided in [35, 38].

## 4 The Case of Monomial Utility Functions

In the following result, we apply Theorems 2 and 3 to characterize the price of anarchy under optimal-congestion-based $\alpha$-SMs of games with monomial utility-functions of fixed degree $p \in (0,1)$ (the proof is deferred to the full version).

**Theorem 4.** . *Given $p \in (0,1)$, a utility-sharing game* SG *with utility functions of type $u(x) = w \cdot x^p$ for some $w > 0$ and an optimal-congestion-based $\alpha$-SM $\Pi_\alpha(\boldsymbol{\sigma}^*)$ for* SG*, we have $\mathrm{PoA}(\mathrm{SG}, \Pi_\alpha(\boldsymbol{\sigma}^*)) \leq \dfrac{1-t}{\alpha} + t^p$, with $t = \min\left\{(\alpha p)^{\frac{1}{1-p}}, 1\right\}$. Furthermore, no $\alpha$-SM can achieve, in general, a better price of anarchy.*

*Remark 3 (PoA without subsidies (cont.)).* For $\alpha = 1$, the tight bounds provided in Corollary 4 are also tight bounds on the price of anarchy of utility-sharing games with monomial profit functions without the use of any $\alpha$-SM, that is, we can re-obtain the tight bounds on the price of anarchy provided in [38].

In the following example we show an application of Theorem 4 with $p = \frac{1}{2}$.

*Example 1.* By applying Theorem 4, we have that the price of anarchy under optimal-congestion-based $\alpha$-SMs of games SG with monomial utility functions of type $u(x) = w \cdot x^{1/2}$ is equal to $\frac{\alpha^2+4}{4\alpha}$ for $\alpha < 2$, and equal to 1 for $\alpha \geq 2$. In Figure 1, we see how the price of anarchy varies over $\alpha \geq 1$, and we compare it with the case of general functions.

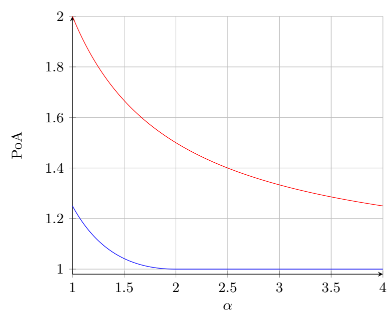


**Fig. 1.** The red line represents, for any $\alpha \geq 1$, the price of anarchy under optimal-based-congestion $\alpha$-SMs for games when the underlying utility functions are not specified (Corollaries 1 and 3), while the blue line represents the price of anarchy of games whose utility functions are monomials of degree 1/2 (a case of Theorem 4). From this comparison, we observe that the bounds on the price of anarchy that depend on the specific class of utility functions (Theorems 2 and 3) may be definitely more precise than the bounds which depend on $\alpha$ only (Corollaries 1 and 3).

## 5 Conclusion and Future Works

In this work, we have shown how to reduce the price of anarchy in a large class of resource-selection games with utility-maximizing players, by resorting to a limited amount of subsidies that can be distributed among resources and is subsequently shared among the players who use them.

Our work leaves several research directions on the problem of improving the efficiency in utility sharing games via limited subsidies.

First of all, our subsidy mechanisms dynamically depend on the actual congestion of each resource. Thus, it would be interesting to show how the efficiency can be improved if subsidies do not depend on the current game configuration. Another research direction could be that of finding subsidy (or taxing) mechanisms with limited budget for more general variants of utility-sharing games, where players can also be cost-minimizers (as in congestion games [48]) and/or have different weights and/or can select different subsets of resources).

Finally, still with the aim of improving the efficiency of the considered games, it would be also interesting to consider other mechanisms than subsidy disbursements (e.g., Stackelberg strategies [49]).

## References

1. Aland, S., Dumrauf, D., Gairing, M., Monien, B., Schoppmann, F.: Exact price of anarchy for polynomial congestion games. SIAM Journal on Computing **40**(5), 1211–1233 (2011)
2. Anshelevich, E., Dasgupta, A., Kleinberg, J.M., Tardos, É., Wexler, T., Roughgarden, T.: The price of stability for network design with fair cost allocation. SIAM Journal on Computing **38**(4), 1602–1623 (2008)
3. Augustine, J., Caragiannis, I., Fanelli, A., Kalaitzis, C.: Enforcing efficient equilibria in network design games via subsidies. Algorithmica **72**(1), 44–82 (2015)
4. Augustine, J., Chen, N., Elkind, E., Fanelli, A., Gravin, N., Shiryaev, D.: Dynamics of profit-sharing games. Internet Math. **11**(1), 1–22 (2015)
5. Awerbuch, B., Azar, Y., Epstein, A.: The price of routing unsplittable flow. In: Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC). pp. 57–66 (2005)
6. Benita, F., Bilò, V., Monnot, B., Piliouras, G., Vinci, C.: Data-driven models of selfish routing: Why price of anarchy does depend on network topology. In: Proceedings of the 16th International Conference on Web and Internet Economics (WINE). pp. 252–265 (2020)
7. Bhawalkar, K., Gairing, M., Roughgarden, T.: Weighted congestion games: price of anarchy, universal worst-case examples, and tightness. ACM Transactions on Economics and Computation **2**(4), 1–23 (2014)
8. Bilò, V.: A unifying tool for bounding the quality of non-cooperative solutions in weighted congestion games. Theory of Computing Systems **62**(5), 1288–1317 (2018)
9. Bilò, V., Monaco, G., Moscardelli, L., Vinci, C.: Nash social welfare in selfish and online load balancing. ACM Trans. Econ. Comput. **10**(2) (2022)
10. Bilò, V., Moscardelli, L., Vinci, C.: Uniform mixed equilibria in network congestion games with link failures. In: Proceedings of the 45th International Colloquium on Automata, Languages and Programming (ICALP). pp. 146:1–146:14 (2018)
11. Bilò, V., Vinci, C.: On the impact of singleton strategies in congestion games. In: Proceedings of the 25th Annual European Symposium on Algorithms (ESA). pp. 17:1–17:14 (2017)
12. Bilò, V., Vinci, C.: Dynamic taxes for polynomial congestion games. ACM Transantions on Economics and Computation **7**(3), 15:1–15:36 (2019)
13. Bilò, V., Vinci, C.: On stackelberg strategies in affine congestion games. Theory of Computing Systems **63**(6), 1228–1249 (2019)
14. Bilò, V., Vinci, C.: The price of anarchy of affine congestion games with similar strategies. Theoretical Computer Science **806**, 641–654 (2020)
15. Bilò, V., Gourvès, L., Monnot, J.: Project games. Theor. Comput. Sci. **940**(Part), 97–111 (2023)
16. Bilò, V., Vinci, C.: Coping with Selfishness in Congestion Games - Analysis and Design via LP Duality. Monographs in Theoretical Computer Science. An EATCS Series, Springer (2023)
17. Bilò, V.: On the robustness of the approximate price of anarchy in generalized congestion games. Theoretical Computer Science **906**, 94–113 (2022)
18. Caragiannis, I., Flammini, M., Kaklamanis, C., Kanellopoulos, P., Moscardelli, L.: Tight bounds for selfish and greedy load balancing. Algorithmica **61**(3), 606–637 (2011)

19. Caragiannis, I., Gkatzelis, V., Vinci, C.: Coordination mechanisms, cost-sharing, and approximation algorithms for scheduling. In: Proceedings of the 13th International Conference on Web and Internet Economics (WINE). pp. 74–87 (2017)
20. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P.: Taxes for linear atomic congestion games. ACM Transactions on Algorithms **7**(1), 13:1–13:31 (2010)
21. Chandan, R., Paccagnan, D., Marden, J.R.: When smoothness is not enough: Toward exact quantification and optimization of the price-of-anarchy. In: 58th IEEE Conference on Decision and Control, CDC 2019, Nice, France, December 11-13, 2019. pp. 4041–4046. IEEE (2019)
22. Chandan, R., Paccagnan, D., Marden, J.R.: Tractable mechanisms for computing near-optimal utility functions. In: AAMAS '21: 20th International Conference on Autonomous Agents and Multiagent Systems, Virtual Event, United Kingdom, May 3-7, 2021. pp. 306–313. ACM (2021)
23. Christodoulou, G., Koutsoupias, E.: The price of anarchy of finite congestion games. In: Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC). pp. 67–73 (2005)
24. Christodoulou, G., Koutsoupias, E., Nanavati, A.: Coordination mechanisms. Theoretical Computer Science **410**(36), 3327–3336 (2009)
25. Cole, R., Correa, J., Gkatzelis, V., Mirrokni, V., Olver, N.: Decentralized utilitarian mechanisms for scheduling games. Games and Economic Behavior **92**, 306–326 (2014)
26. Cole, R., Dodis, Y., Roughgarden, T.: How much can taxes help selfish routing? Journal of Computer and System Sciences **72**(3), 444–467 (2006)
27. Correa, J., de Jong, J., de Keijzer, B., Uetz, M.: The inefficiency of nash and subgame perfect equilibria for network routing. Mathematics of Operations Research **44**(4), 1286–1303 (2019)
28. von Falkenhausen, P., Harks, T.: Optimal cost sharing for resource selection games. Mathematics of Operations Research **38**(1), 184–208 (2013)
29. Ferguson, B.L., Brown, P.N., Marden, J.R.: The effectiveness of subsidies and taxes in atomic congestion games. IEEE Control Systems Letters **6**, 614–619 (2022)
30. Fotakis, D.: Stackelberg strategies for atomic congestion games. Theory of Computing Systems **47**(1), 218–249 (2010)
31. Gkatzelis, V., Kollias, K., Roughgarden, T.: Optimal cost-sharing in general resource selection games. Operations Research **64**(6), 1230–1238 (2016)
32. Goemans, M.X., Li, L.E., Mirrokni, V.S., Thottan, M.: Market sharing games applied to content distribution in ad hoc networks. IEEE J. Sel. Areas Commun. **24**(5), 1020–1033 (2006)
33. Gollapudi, S., Kollias, K., Panigrahi, D., Pliatsika, V.: Profit sharing and efficiency in utility games. In: 25th Annual European Symposium on Algorithms, ESA 2017, September 4-6, 2017, Vienna, Austria. LIPIcs, vol. 87, pp. 43:1–43:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2017)
34. Kleer, P.: Price of anarchy for parallel link networks with generalized mean objective. OR Spectr. **45**(1), 27–55 (2023)
35. Kleinberg, J.M., Oren, S.: Mechanisms for (mis)allocating scientific credit. Algorithmica **84**(2), 344–378 (2022)
36. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science (STACS). pp. 404–413 (1999)
37. Lücking, T., Mavronicolas, M., Monien, B., Rode, M.: A new model for selfish routing. Theoretical Computer Science **406**(3), 187–2006 (2008)

38. Marden, J.R., Roughgarden, T.: Generalized efficiency bounds in distributed resource allocation. IEEE Trans. Autom. Control. **59**(3), 571–584 (2014)
39. Marden, J.R., Wierman, A.: Distributed welfare games. Oper. Res. **61**(1), 155–168 (2013)
40. Nash, J.F.: Equilibrium points in $n$-person games. Proceedings of the National Academy of Science **36**(1), 48–49 (1950)
41. Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V.V. (eds.): Algorithmic Game Theory. Cambridge University Press (2007)
42. Paccagnan, D., Chandan, R., Ferguson, B.L., Marden, R.J.: Optimal taxes in atomic congestion games. ACM Transactions on Economics and Computation **9**(3), 19:1–19:33 (2021)
43. Paccagnan, D., Gairing, M.: In congestion games, taxes achieve optimal approximation. In: Proceedings of the 22nd ACM Conference on Economics and Computation (EC). pp. 743–744 (2021)
44. Paccagnan, D., Chandan, R., Marden, J.R.: Utility design for distributed resource allocation - part I: characterizing and optimizing the exact price of anarchy. IEEE Trans. Autom. Control. **65**(11), 4616–4631 (2020)
45. Paccagnan, D., Chandan, R., Marden, J.R.: Utility and mechanism design in multiagent systems: An overview. Annu. Rev. Control. **53**, 315–328 (2022)
46. Paccagnan, D., Marden, J.R.: Utility design for distributed resource allocation - part II: applications to submodular, covering, and supermodular problems. IEEE Trans. Autom. Control. **67**(2), 618–632 (2022)
47. Ravindran Vijayalakshmi, V., Skopalik, A.: Improving approximate pure nash equilibria in congestion games. In: Proceedings of the 16th International Conference of Web and Internet Economics (WINE). pp. 280–294 (2020)
48. Rosenthal, R.W.: A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory **2**(1), 65–67 (1973)
49. Roughgarden, T.: Stackelberg scheduling strategies. SIAM Journal on Computing **33**(2), 332–350 (2004)
50. Roughgarden, T.: Intrinsic robustness of the price of anarchy. Journal of ACM **62**(5), 32:1–32:42 (2015)
51. Roughgarden, T., Tardos, E.: How bad is selfish routing? Journal of ACM **49**(2), 236–259 (2002)
52. Stein, N., Tamir, T.: Stackelberg strategies for weighted load balancing games. In: Proceedings of the 17th Conference on Computer Science and Intelligence Systems, FedCSIS 2022, Sofia, Bulgaria, September 4-7, 2022. pp. 373–382 (2022)
53. Vetta, A.: Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In: Proceedings of the 43rd Symposium on Foundations of Computer Science (FOCS). pp. 416–425 (2002)